10.5 Implicit Functions.

A. Examples.

1. Consider the equation $x^2 + y^2 = 1$. Does there exist a function $g(x)$ such that for all $x \in [-1,1]$, $x^2 + g(x)^2 = 1$? In other words, can we solve this equation for y in terms of x?

   Answer: Clearly no in general, but we can say the following: Given a point (x_0, y_0) on the curve for which $\frac{dy}{dx} = -\frac{x}{y}$ is defined (that is, if $y_0 \neq 0$), there is a small open interval (a, b) containing x_0 such that such a $g(x)$ exists for $x \in (a, b)$. Also note that if $y_0 = 0$ then no such interval or function exists.

2. Suppose we have a curve in the plane given by $f(x, y) = 0$.

$$\frac{dy}{dx} = {-\frac{\partial f}{\partial x}}\Big/{\frac{\partial f}{\partial y}}$$

so the curve has a nonvertical tangent exactly when $\frac{\partial f}{\partial y} \neq 0$ and we should be able to solve for y in terms of x. But how?

3. Define $F: \mathbb{E}^2 \to \mathbb{E}^2$ by $F(x, y) = (x, f(x, y))$.

B. Theorem (10.5.1) Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^m)$ where D is an open subset of $\mathbb{E}^{n+m}$ and that for some point $(\mathbb{x}_0, \mathbb{y}_0) \in D$, $\mathbb{f}(\mathbb{x}_0, \mathbb{y}_0) = \mathbb{0}$ and

$$\frac{\partial(f_1, \dots, f_m)}{\partial(y_1, \dots, y_m)}(\mathbb{x}_0, \mathbb{y}_0) \neq 0$$

Then there is an open set $U \subseteq \mathbb{E}^n$ containing $\mathbb{x}_0$ and a function $\mathbb{g} \in C^1(U, \mathbb{E}^m)$ such that for all $\mathbb{x} \in U$, $\mathbb{f}\big(\mathbb{x}, \mathbb{g}(\mathbb{x})\big) = \mathbb{0}$.

4. Example. $f: \mathbb{E}^2 \to \mathbb{E}^2$ given by

$$f(x, y) = (x^2 + y^2, x + y)$$

5. <u>Definition</u>. (Local invertibility) A function $\mathbb{f}\colon \mathbb{E}^n \to \mathbb{E}^n$ is *locally one-to-one* in an open set V if for every $\mathbb{x}_0 \in V$, there is an $\epsilon > 0$ such

that $\mathbb{f}$ restricted to $B(\mathbb{x}_0, \epsilon)$ is one-to-one. If $\mathbb{f}$ is one-to-one on a set E then we say $\mathbb{f}$ is *globally one-to-one* on E.

6. Example. Let $f(x, y) = (x \cos y, x \sin y)$ be defined on the open set $V = \{(x, y): x > 0\}$. Then f is locally $1 - 1$ on V but not globally $1 - 1$ on V.

C. The Jacobian.

1. Definition. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$ is in $C^1(D, \mathbb{E}^n)$. Then the *Jacobian* of $\mathbb{f}$ at $\mathbb{x} \in D$ is given by $\det(\mathbb{f}'(\mathbb{x}))$.

2. Theorem. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$, D an open subset of $\mathbb{E}^n$, is in $C^1(D, \mathbb{E}^n)$, and suppose that $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$. Then $\mathbb{f}$ is locally one-to-one in D.

D. Inverse Function Theorem.

1. <u>Lemma</u>. (Open Mapping Theorem, Thm. 10.4.2). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$, then $\mathbb{f}$ is an open mapping, that is, $\mathbb{f}$ maps open subsets of D to open subsets of $\mathbb{E}^n$.

2. <u>Theorem</u> (10.4.3). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all

$\mathbb{x} \in D$, and if $\mathbb{f}$ is globally one-to-one on D, then $\mathbb{f}^{-1} \in C^1(f(D), \mathbb{E}^n)$ and

$$(\mathbb{f}^{-1})'\big(\mathbb{f}(\mathbb{x})\big) = (\mathbb{f}'(x))^{-1}$$