10.4 Inverse Functions.

A. Local Invertibility.

1. Example. Let $f(x) = x^2$. For any $x_0 \in (0, \infty)$ there is an open interval around x_0 such that f restricted to that interval is injective. Similarly for any $x_0 \in (-\infty, 0)$. However f is not injective on any open interval containing 0.

2. <u>Remark</u>. Suppose $f:(a,b)\to\mathbb{R}$ is C^1on (a,b) and that for some $c\in(a,b)$, $f^{'}(c)\neq 0$. Then the following conclusions hold:

   a. $f^{'}(x)\neq 0$ on some interval $(c-\delta, c+\delta)$.
   b. $f(x)$ is monotone on this interval, hence invertible.
   c. f^{-1} is also C^1on some open interval around $f(c)$, and $(f^{-1})^{'}(y) = 1/f^{'}(f^{-1}(y))$ for y in this interval.

3. <u>Example</u>. $f: \mathbb{E}^2 \to \mathbb{E}^2$ given by
$f(x, y) = (x^2 + y^2, x + y)$

4. <u>Definition</u>. (Local invertibility) A function $\mathbb{f}: \mathbb{E}^n \to \mathbb{E}^n$ is *locally one-to-one* in an open set V if for every $\mathbb{x}_0 \in V$, there is an $\epsilon > 0$ such that $\mathbb{f}$ restricted to $B(\mathbb{x}_0, \epsilon)$ is one-to-one. If $\mathbb{f}$ is one-to-one on a set E then we say $\mathbb{f}$ is *globally one-to-one* on E.

5. <u>Example</u>. Let $f(x, y) = (x \ \cos y\,, x \ \sin y)$ be defined on the open set $V = \{(x, y): x > 0\}$. Then f is locally $1 - 1$ on V but not globally $1 - 1$ on V.

B. The Jacobian.

1. <u>Definition</u>. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$ is in $C^1(D, \mathbb{E}^n)$. Then the *Jacobian* of $\mathbb{f}$ at $\mathbb{x} \in D$ is given by $\det(\mathbb{f}'(\mathbb{x}))$.

2. <u>Theorem</u>. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$, D an open subset of $\mathbb{E}^n$, is in $C^1(D, \mathbb{E}^n)$, and suppose that $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$. Then $\mathbb{f}$ is locally one-to-one in D.

C. Inverse Function Theorem.

1. Lemma. (Open Mapping Theorem, Thm. 10.4.2). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$, then $\mathbb{f}$ is an open mapping, that is, $\mathbb{f}$ maps open subsets of D to open subsets of $\mathbb{E}^n$.

2. <u>Theorem</u> (10.4.3). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$, and if $\mathbb{f}$ is globally one-to-one on D, then $\mathbb{f}^{-1} \in C^1(f(D), \mathbb{E}^n)$ and

$$(\mathbb{f}^{-1})'\big(\mathbb{f}(\mathbb{x})\big) = (\mathbb{f}'(x))^{-1}$$