10.3 The Chain Rule.

A. The Product Rule (Section 10.2).

1. Theorem. Let $\mathbb{f}, \mathbb{g}: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ be differentiable at $\mathbb{x}_0 \in D$. Then
$$(\mathbb{f} \cdot \mathbb{g})'(\mathbb{x}_0) = \mathbb{f}(\mathbb{x}_0)\mathbb{g}'(\mathbb{x}_0) + \mathbb{g}(\mathbb{x}_0)\mathbb{f}'(\mathbb{x}_0)$$

2. Remark. How do we interpret this formula in terms of linear transformations?

3. Proof of Theorem:

B. The Chain Rule.

1. <u>Theorem</u>. Suppose that $\mathbb{g}: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ and $\mathbb{f}: V \subseteq \mathbb{E}^m \to \mathbb{E}^p$, where D is an open subset of $\mathbb{E}^n$ and V is an open subset of $\mathbb{E}^m$ such that $\mathbb{g}(D) \subseteq V$, and that $\mathbb{g}'(\mathbb{x}_0)$ and $\mathbb{f}'(\mathbb{g}(\mathbb{x}_0))$ both exist at $\mathbb{x}_0 \in D$. Then
$$(\mathbb{f} \circ \mathbb{g})'(\mathbb{x}_0) = \mathbb{f}'(\mathbb{g}(\mathbb{x}_0))\, \mathbb{g}'(\mathbb{x}_0)$$

2. <u>Remark</u>. How do we interpret this theorem in terms of linear transformations?

3. Proof of Theorem.

C. The Mean Value Theorem.

1. <u>Theorem</u>. Let $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

2. <u>Remark</u>. A natural generalization to functions $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ might be: Suppose that $\mathbb{f}: V \to \mathbb{E}^m$ where V is a ball in $\mathbb{E}^n$. Then given $\mathbb{a}, \mathbb{b} \in V$ there is a $\mathbb{c}$ on the line segment joining $\mathbb{a}$ and $\mathbb{b}$ such that $\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a}) = \mathbb{f}'(\mathbb{c})(\mathbb{b} - \mathbb{a})$.

3. Note first of all that the dimensions of the matrices work out, but the theorem does not hold.

4. For $\mathbb{f}$ as above, consider the function $\mathbb{g}: \mathbb{R} \to \mathbb{E}^m$ given by $\mathbb{g}(t) = t\mathbb{b} + (1-t)\mathbb{a}$. Then look at the function $\mathbb{f} \circ \mathbb{g}: \mathbb{R} \to \mathbb{E}^m$. What can we say in this case?

5. <u>Theorem</u>. (MVT 1) Let $V \subseteq \mathbb{E}^n$ be open and convex, and let $\mathbb{f}: V \to \mathbb{E}^m$ be differentiable on V. Let $\mathbb{a}$, $\mathbb{b} \in V$ and let $\mathbb{u} \in \mathbb{E}^m$ be an arbitrary vector. Then there is a $\mathbb{c}$ on the line segment joining $\mathbb{a}$ and $\mathbb{b}$ such that

$$\mathbb{u} \cdot \big(\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a})\big) = \mathbb{u} \cdot (\mathbb{f}'(\mathbb{c})(\mathbb{b} - \mathbb{a}))$$

6. <u>Example</u>. Let $f(x, y) = x(y - 1)$. Then $f(1,1) - f(0,0) = 0$, and $\nabla f(x, y)$ does not vanish on the line segment joining $(0,0)$ and $(1,1)$.

7. Proof of MVT 1.

8. Theorem. (MVT 2) Under the hypotheses of the previous theorem, there exist vectors $\mathbb{c}_1, \mathbb{c}_2, \ldots, \mathbb{c}_m \in V \subseteq \mathbb{E}^n$ such that

$$\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a}) = \left[\frac{\partial f_i}{\partial x_j}\left(\mathbb{c}_j\right)\right](\mathbb{b} - \mathbb{a})$$