$$f(x) = \underset{m}{f(a)} + f'(a)(x-a) + \underbrace{o(|x-a|)}_{} \leftarrow R(x-a)$$

$$\Updownarrow \qquad \lim_{x\to a} \frac{R(x-a)}{x-a} = 0.$$

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{\vec{h}\to 0} \frac{\|f(\vec{x}+\vec{h}) - f(\vec{x}) - A\vec{h}\|}{\|\vec{h}\|} = 0. \quad \leftarrow \varepsilon(h)$$

$$\Updownarrow$$

$$\lim \frac{\|\varepsilon(\vec{h})\|}{\|\vec{h}\|} = 0.$$

Define $\Delta f = f(\vec{x}+\vec{h}) - f(\vec{x}) = A\vec{h} + \varepsilon(\vec{h})$.

Collection of $n\times n$ Matrices

$\det A \neq 0 \Leftrightarrow A$ is nonsingular

$\det : M \longrightarrow \mathbb{R}$

$$A^{-1} = \left[\frac{\det A_{ij}}{\det A}\right]_{ij}$$

$\det(I) = 1$

$\det(cA) = c^n \det(A)$

permute rows/columns $\Rightarrow \det(A') = -\det(A)$

$\det(AB) = \det(A)\det(B)$

Change of Variables:

$$\int_{g(U)} f(t)\,dt = \int_U f(g(x))\,|\det g'(x)|\,dx$$

Know: ~~If G is a cube~~ Given $\varepsilon > 0$ there is a $\delta > 0$ s.t. if C is a cube of side $< \delta$ then for all $x \in C$

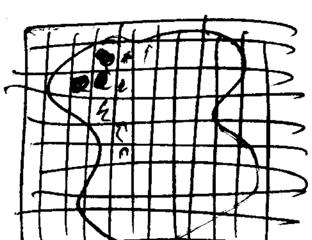
$$v(g(C)) \leq (|\det g'(x)| + c\varepsilon)\, v(C)$$

where c does not depend on x.

Lemma: Given U bounded, open Jordan region, $g \in C^1(\bar{U}, E^n)$

$$v(g(U)) \leq \int_U |\det g'(x)|\,dx$$

Proof: Approximate U from inside by small cubes in a grid. For all such C,



$$v(g(C)) \leq (|\det g'(\bar{x})| + \varepsilon c)\, v(C)$$

So

$$\nu\Big(g\big(\bigcup_j C_j\big)\Big) = \nu\Big(\bigcup_j g(C_j)\Big)$$

$$\le \sum_j \nu(g(C_j)) \le \sum_j \inf_{\vec{x} \in C_j} |\det g'(\vec{x})| \, \nu(C_j)$$

$$+ \, c\varepsilon \sum_j \nu(C_j)$$

$$\le L(|\det g'|, \mathcal{P}) + c\varepsilon \nu(U).$$

Need one more Lemma: does $g(\bigcup_j C_j)$ approximate $g(U)$? Turns out to be fine.

Finally we arrive at

$$\nu(g(U)) \le L(|\det g'|, \mathcal{P}) + c\varepsilon\nu(U).$$

$$\le \int_U |\det g'(\vec{x})| \, d\vec{x} + c\varepsilon\nu(U)$$

Let $\varepsilon \to 0$.

Lemma: If f is integrable on $g(\mathcal{U})$ then

$$\int_{g(\mathcal{U})} f(t)\,dt \le \int_{\mathcal{U}} f(g(\vec{x}))\,|\det g'(\vec{x})|\,d\vec{x}.$$

Pf: Approximate $g(\mathcal{U})$ (a Jordan Region) from inside by rectangles in a grid P, call them R_j and let $\mathcal{U}_j = g^{-1}(R_j)$. Then letting $m_j = \inf\{f(t) : t \in R_j\}$, ~~Then~~

$m_j = \inf\{f(g(x)) : x \in g^{-1}(R_j)\}$. So

$$m_j\,v(R_j) = m_j\,v(g(\mathcal{U}_j)) \le m_j \int_{\mathcal{U}_j} |\det g'(\vec{x})|\,d\vec{x}$$

$$= \int_{\mathcal{U}_j} m_j\,|\det(g'(\vec{x}))|\,d\vec{x} \le \int_{\mathcal{U}_j} f(g(x))\,|\det g'(\vec{x})|\,d\vec{x}$$

Summing over j gives

$$L(f, P) \le \int_{(\bigcup_j \mathcal{U}_j)} f(g(\vec{x}))\,|\det g'(\vec{x})|\,d\vec{x}$$

$$= \int_{\mathcal{U}} f(g(\vec{x}))\,|\det g'(\vec{x})|\,d\vec{x}$$

Taking sup over P gives.

$$\int_{g(U)} f(t)\,dt \le \int_U f(g(\vec{x}))\,|\det g'(\vec{x})|\,d\vec{x}.$$

To complete proof, we need the other inequality.

Let $W = g(U)$ then $U = g^{-1}(W)$. Apply previous result with g^{-1} replacing g: and with $F(x) = f(g(x))\,|\det g'(x)|$ for $x \in U = g^{-1}(W)$. Then

$$\int_{g^{-1}(W)} F(\vec{x})\,d\vec{x} \le \int_W F(g^{-1}(t))\,|\det (g^{-1})'(t)|\,dt$$

By chain rule $(g^{-1})'(t) = [g'(g^{-1}(t))]^{-1}$

and $F(g^{-1}(t)) = f(t)\,|\det g'(g^{-1}(t))|$. Hence.

$$F(g^{-1}(t))\,|\det (g^{-1})'(t)|\,|\det g'(g^{-1}(t))|$$

$$= f(t)\,|\det [g'(g^{-1}(t))]^{-1}\cdot \det g'(g^{-1}(t))|$$

$$= f(t).$$

Therefore,

$$\int_U f(g(x))\,|\det g'(x)|\,dx \le \int_{g(U)} f(t)\,dt.$$

And we are done!

$(g^{-1})'(t)$

$$g(g^{-1}(t)) = t$$

$$g'(g^{-1}(t))(g^{-1})'(t) = I$$

$$(g^{-1})'(t) = \left[g'(g^{-1}(t))\right]^{-1}$$