Change of variables in 1-dim.

$$\int_{g([a,b])} f(t)\,dt = \int_{[a,b]} f(g(x))\,|g'(x)|\,dx$$

Idea of proof:

① Prove when $f \equiv 1$, i.e.

$$\ell(g([a,b])) = \int_a^b |g'(x)|\,dx$$

② Prove this locally, i.e. for small intervals

Result: Suppose $g \in C^1[a,b]$ and $g' \neq 0$ on $[a,b]$. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $[\alpha,\beta] \subseteq [a,b]$ with $|\beta-\alpha| < \delta$, and for all $t \in [\alpha,\beta]$

$$|g(\beta)-g(\alpha)| \leq |g'(t)|\,|\beta-\alpha| + \varepsilon|\beta-\alpha| \quad \text{and}$$

$$|g(\beta)-g(\alpha)| \geq |g'(t)|\,|\beta-\alpha| - \varepsilon|\beta-\alpha|$$

[Comes from:

$$g(y) = g(t) + g'(t)(y-t) + o(|y-t|)$$

$\forall\ |y-t|$ small enough.

Let $y=\beta$ and $y=\alpha$.]

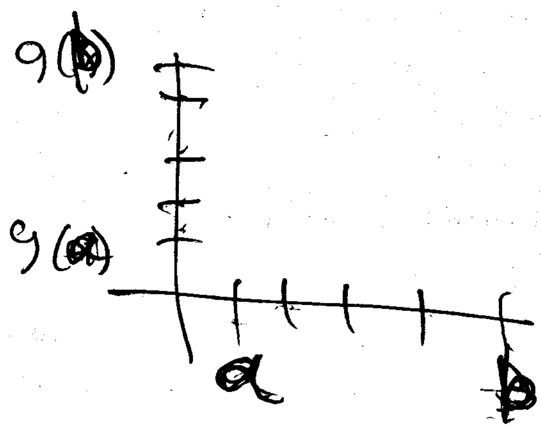
Key: δ does not depend on t or $[\alpha, \beta]$.

③ How do we prove ①?
Fix $\varepsilon > 0$ and choose a partition P of $[a,b]$ such that $|x_i - x_{i-1}| < \delta$ all i. For each i,

$$|g(x_i) - g(x_{i-1})| \le |g'(t)|\,|x_i - x_{i-1}| + \varepsilon |x_i - x_{i-1}|$$

Hence
$$|g(x_i) - g(x_{i-1})| \le \left(\inf_{t \in [x_{i-1}, x_i]} |g'(t)|\right) |x_i - x_{i-1}| + \varepsilon |x_i - x_{i-1}|$$

Since $g([x_{i-1}, x_i])$ is an interval of length $|g(x_i) - g(x_{i-1})|$,



$$\ell(g([a,b])) \le L(|g'|, P) + \varepsilon |b-a|$$
$$\le \int_a^b |g'(t)|\,dt + \varepsilon |b-a|$$

let $\varepsilon \to 0$ $\quad \ell(g([a,b])) \le \int_a^b |g'(t)|\,dt$

Similarly, use

$$|g(x_i)-g(x_{i-1})| \ge |g'(t)|\,|x_i-x_{i-1}| - \varepsilon|x_i-x_{i-1}|$$

to get other inequality.

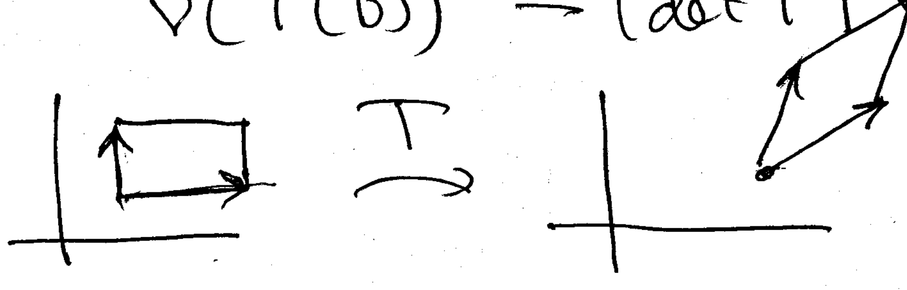
What about E^n?

Ultimate result: Given bounded open set $U \subseteq E^n$ and $g: U \to E^n$, $g \in C^1(\overline{U}, E^n)$ then for every Riemann integrable f on $g(U)$,

$$\int_{g(U)} f(t)\,dt = \int_{U} f(g(x))\,|\det g'(x)|\,dx$$

Basic fact: If $B \subseteq E^n$ is a rectangle and $T \in \mathcal{L}(E^n)$, then $T(B)$ is a parallelogram in E^n and

$$v(T(B)) = |\det T|\,v(B)$$



Main Lemma: Given $\varepsilon > 0$ there is a $\delta > 0$ such that for every cube $C \subseteq U$ whose sides are $< \delta$ in length and every $\vec{x} \in C$

$$\nu(g(C)) \leq \left(|\det g'(\vec{x})| + c\varepsilon \right) \nu(C)$$

where c is a constant depending only on n and on g.

Proof: ① First of all, $g \in C'(\overline{U}, \mathbb{R}^n)$ and $\overline{U}$ is closed and bounded, hence compact

$\therefore$ (a) $\|g'(\vec{z})\|$ is bounded on $\overline{U}$, hence by MVT there is an M such that for all $\vec{x}, \vec{y}$ in $\overline{U}$, $\|g(\vec{x}) - g(\vec{y})\| \leq M \|\vec{x} - \vec{y}\|$. That is, distances are magnified by a fixed factor.

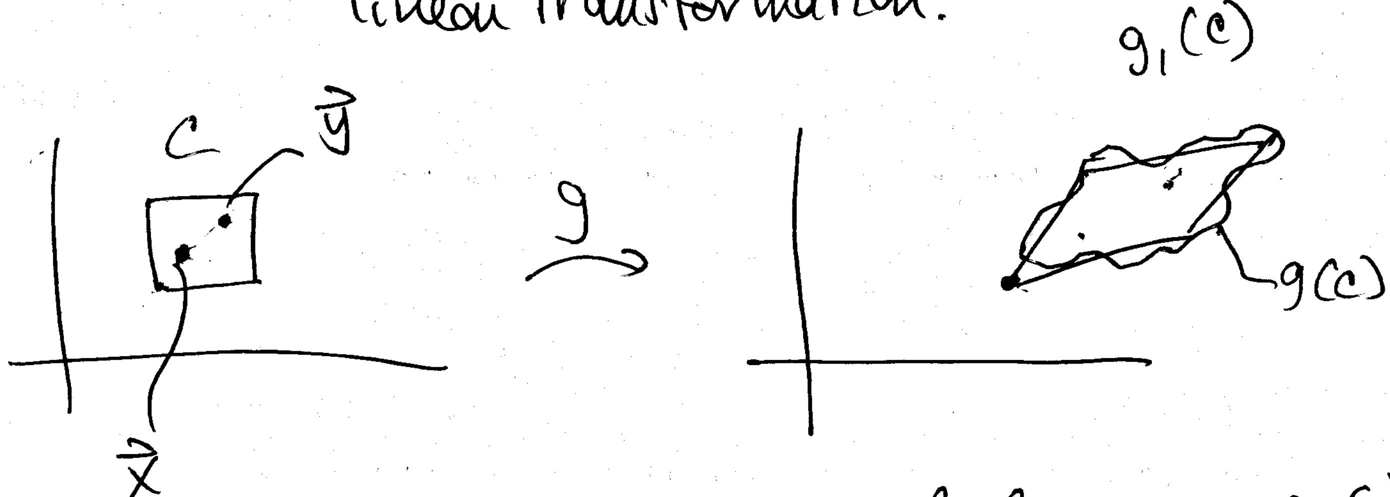
(b) g' is uniformly continuous on $\overline{U}$. This implies that for any $\vec{x}, \vec{y} \in \overline{U}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\vec{x} - \vec{y}\| < \delta$ then

$g(\vec{y}) = g(\vec{x}) + g'(\vec{x})(\vec{y} - \vec{x}) + R(\vec{y} - \vec{x})$ where $\|R(\vec{y} - \vec{x})\| < \varepsilon \|\vec{y} - \vec{x}\|$. The δ does not depend on $\vec{x}$ and $\vec{y}$.

② In 1-dim, g(interval) = another interval. Not so simple in E^n. How does g distort a rectangle or cube. Let C be a cube of side $< \delta$ and $\vec{x}$ any point in C.

$$g(\vec{y}) = g(\vec{x}) + g'(\vec{x})(\vec{y}-\vec{x}) + R(\vec{y}-\vec{x})$$
$$= \underbrace{g_1(\vec{x})} + R(\vec{y}-\vec{x})$$

↑ affine
linear transformation:



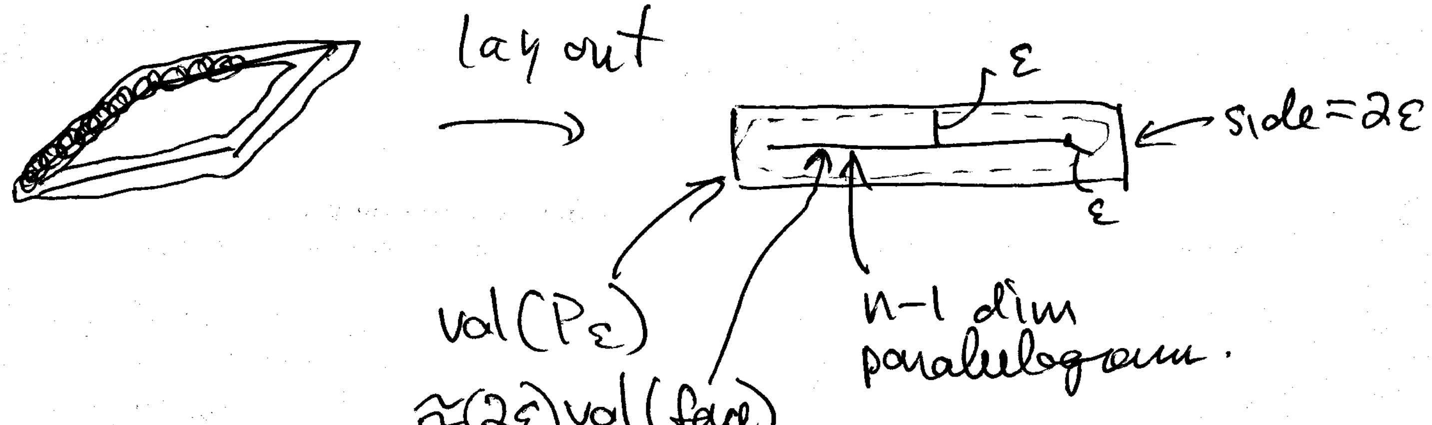
So g takes C to a parallelogram $g_1(C)$ + small perturbation no larger than $\varepsilon\|\vec{y}-\vec{x}\|$

We can write $g(C) = g_1(C) \cup P_\varepsilon$

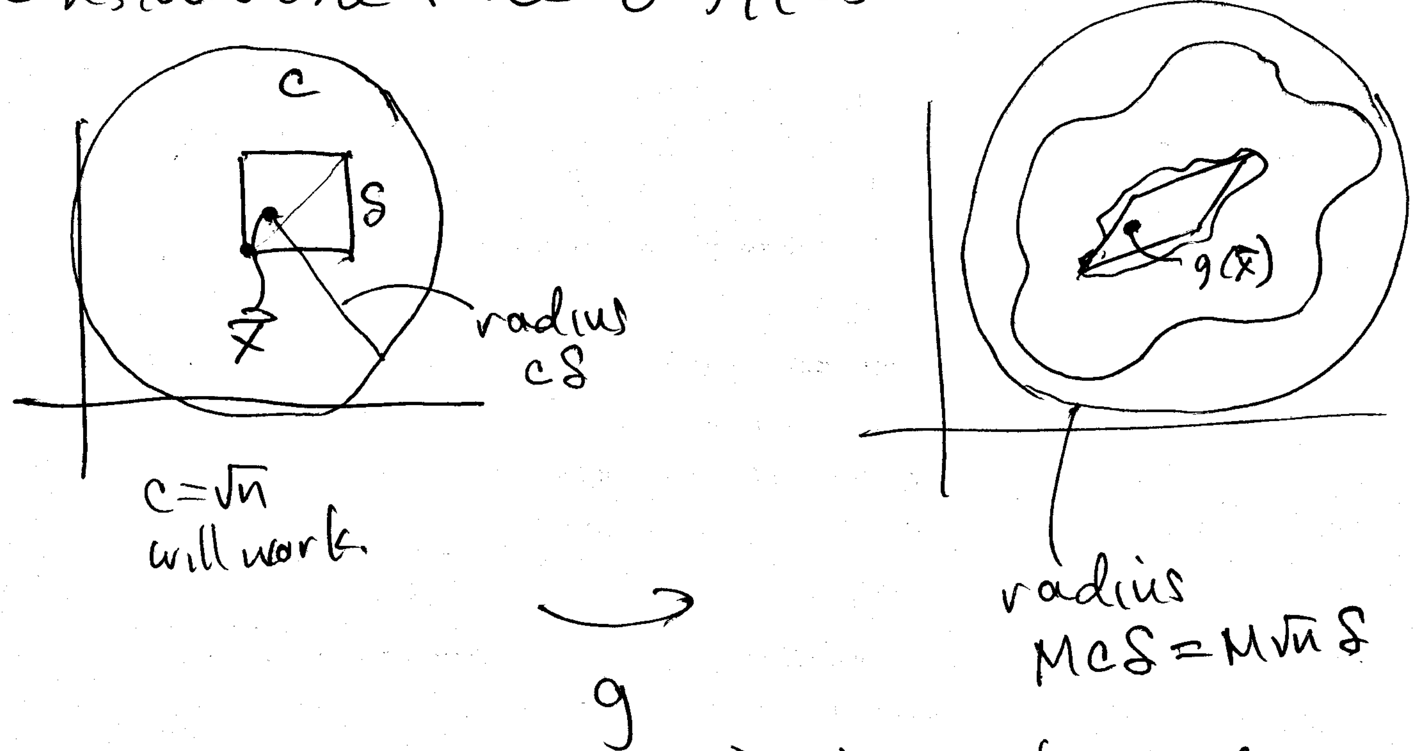
where P_ε is contained in the union of small balls around each point on the "perimeter" of $g_1(C)$.

By perimeter of a parallelogram in $\mathbb{E}^n$ we mean the union of the "faces" of the parallelogram which are $(n-1)$ dim parallelograms. There are $2n$ of these.

③ Estimate the volume of P_ε.



Consider one face of $g_1(C)$:



So every face of $g_1(C)$ is contained in a rectangle whose side length is no more than $2M\sqrt{n}\,\delta$.

n-dim volume is

$$< (2\sqrt{n}\,\varepsilon\delta)(c\delta + 2\sqrt{n}\,\varepsilon\delta)^{n-1} = c_0\,\varepsilon\cdot\delta^n$$
$$= c_0\cdot\varepsilon\,v(C).$$

Noting that $v(g_1(C)) = |\det g'(\vec{x})|\,v(C)$ we arrive at

$$v(g(C)) \le |\det g'(\vec{x})|\,v(C) + c_0\varepsilon\,v(C)$$
$$= (|\det g'(\vec{x})| + c_0\varepsilon)\,v(C).$$