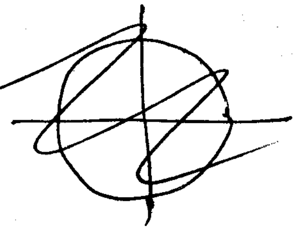
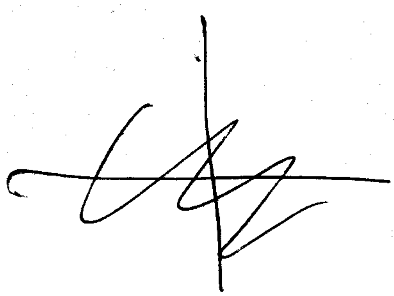
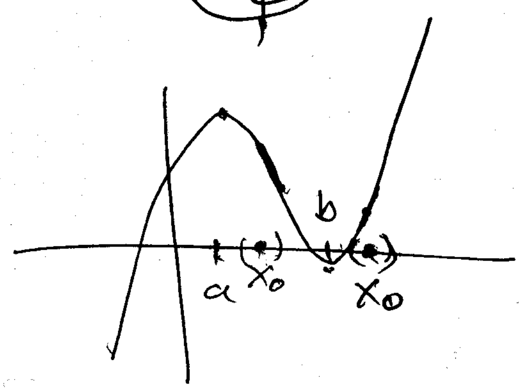
$f: D \subseteq E^n \longrightarrow E^m$

f is locally inj on D if for every $\vec{x}_0 \in D$ there is an $\varepsilon > 0$ such that $f\big|_{B(\vec{x}_0, \varepsilon)}$ is injective.

~~eg $x^2+y^2=1$~~







$f(x)$

f is not globally injective on $\mathbb{R}$

$f(x)$ is locally inj on $\mathbb{R} \setminus \{a, b\}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \longmapsto (x \sin y, x \cos y)$

Not globally inj on $\mathbb{R}^2$

but is locally inj on $\mathbb{R}^2$

C. Integration over Jordan Regions.

1. <u>Definition</u> 5. Let $S \subseteq \mathbb{E}^n$ be a Jordan region, $f: S \to \mathbb{E}^1$ be bounded, and suppose that $S \subseteq B$ for some rectangle B. Extend f to a function on B by letting $f(x) = 0$ if $x \in B \setminus S$.

   The *upper sum* of f with respect to a grid $\mathcal{P}$ of B is

$$U(f, \mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap \cancel{E} \neq \emptyset}} M_j \, v(B_j)$$

   where $M_j = \sup\{f(\mathbb{x}) : \mathbb{x} \in B_j\}$ and the *lower sum* by

$$L(f, \mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap \cancel{E} \neq \emptyset}} m_j \, v(B_j)$$

   where $m_j = \inf\{f(\mathbb{x}) : \mathbb{x} \in B_j\}$.

2. Definition 6. The upper and lower integrals of f over S are given by

$$\overline{\int_S} f = \sup\{L(f,\mathcal{P})\colon \mathcal{P} \text{ a grid on } B\}$$

and

$$\underline{\int_S} f = \inf\{U(f,\mathcal{P})\colon \mathcal{P} \text{ a grid on } B\}$$

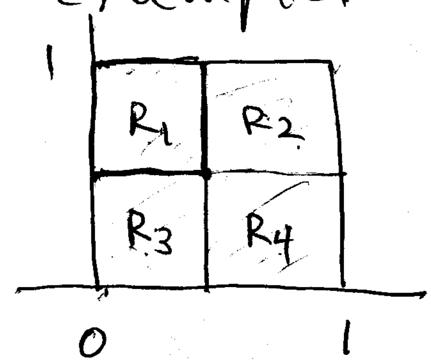
if $\overline{\int_S} f = \underline{\int_S} f$ then the common value is the Riemann integral of f on S and is denoted $\int_S f$.

3. Remark.
   a. The value of the integral over S is independent of the choice of rectangle B containing S.

   b. What is the point of defining the integral this way, i.e., by enclosing S in a rectangle?

On $\mathbb{R}$, assumption is f is defined on $[a,b]$. In E^n we assume that region of integration is contained is some rectangle B

Example:



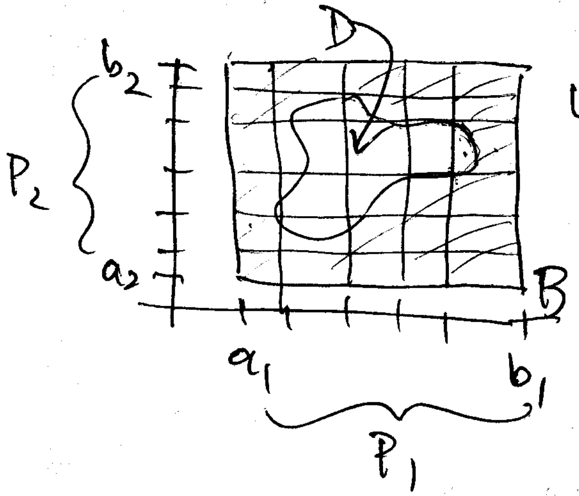
$$f(x) = \begin{cases} 1 & \text{on } R_1^{o} \\ -1 & \text{on } \partial R_1 \end{cases}$$ (boundary of R_1)

$(\partial E = \overline{E} \setminus E^{o})$

Integration in $\mathbb{E}^n$

$f : D \subseteq \mathbb{E}^n \to \mathbb{E}^1 = \mathbb{R}$

Define $\int_D f(x)\,dx$



Let $f = 0$ outside D

$\sup\{f(x) : x \in B_j\}$

$$U(f, P) = \sum_{\substack{B_j \in P \\ B_j \cap \bar{D} \neq \emptyset}} M_j \, v(B_j)$$

$$L(f, P) = \sum_{\substack{B_j \in P \\ B_j \cap D^\circ \neq \emptyset}} m_j \, v(B_j)$$

$\inf\{f(x) : x \in B_j\}.$

$P = P_1 \times P_2$

also denote by P the subrectangles in grid

---

look at $\sup_P L(f, P) \leq \inf_P U(f, P)$

If equal, common value is $\int_D f(x)\,dx$.

or $\int_D f$

Suppose we don't place $\overline{R_1}$ in a large rectangle and extend f to be zero.
dom(f)

Suppose we just take

$$M_j = \sup\{f(x) : x \in B_j \cap \overline{R_1}\}$$

$$m_j = \inf\{f(x) : x \in B_j \cap \overline{R_1}\}.$$

Consider a partition of $[0,1]\times[0,1] = B$ given by $P = \{R_1, R_2, R_3, R_4\}$. ← all closed

$$U(f,P) = \sum_j M_j\, v(B_j)$$

$$= (1)\left(\tfrac{1}{4}\right) + (-1)\left(\tfrac{1}{4}\right) + (-1)\left(\tfrac{1}{4}\right) + (-1)\left(\tfrac{1}{4}\right) = -\tfrac{1}{2}$$

But we want $\int_{R_1} f = \frac{1}{4}$

But $U(f,P) = -\frac{1}{2} < \frac{1}{4} = \int_{R_1} f$ NO GOOD

What if we extend f to be 0 outside $\overline{R_1}$?

$$U(f, P) = (1)\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{4}\right)$$
$$= \frac{1}{4}$$

Change of Variables Formula.

<u>One-dimension:</u>

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

$a \mapsto g(a)$

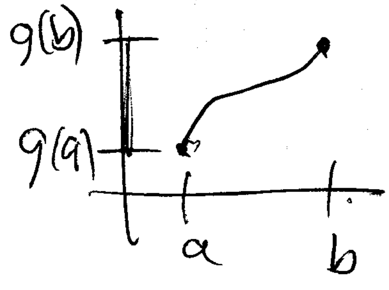
$b \mapsto g(b)$

① Assume: ⓐ g is C^1 on $[a,b]$

ⓑ $g' \neq 0$ on $[a,b]$. This means either $g' > 0$ or $g' < 0$ on $[a,b]$

$g' > 0 \Longrightarrow g(a) < g(b)$

So $\displaystyle\int_{g(a)}^{g(b)} f(u)\,du = \int_{g([a,b])} f(u)\,du.$



$g' < 0 \Longrightarrow g(b) < g(a)$

$$\int_{g(a)}^{g(b)} f(u)\,du = -\int_{g(b)}^{g(a)} f(u)\,du = -\int_{g([a,b])} f(u)\,du$$

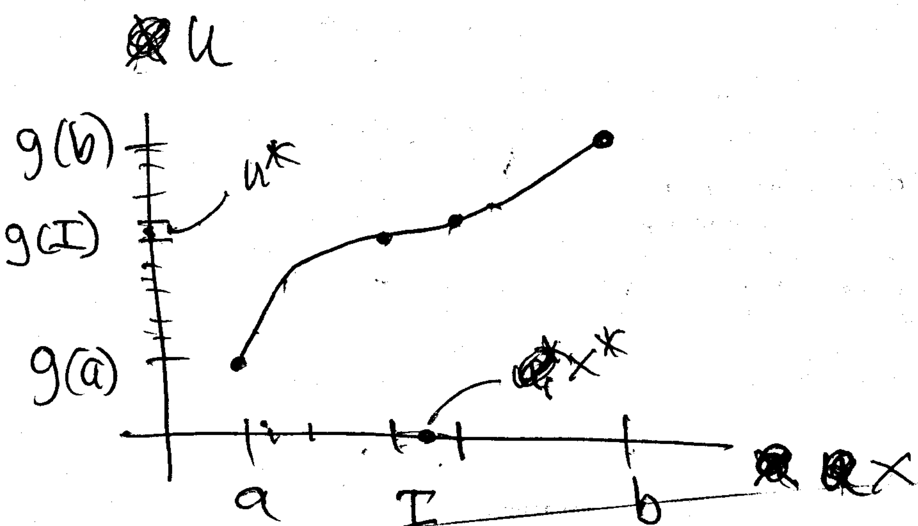
Therefore change of variables becomes.

$$\int_{[a,b]} f(g(x))\,|g'(x)|\,dx = \int_{g([a,b])} f(u)\,du.$$

~~② Why does this work?~~

OR

$$\int_{[a,b]} f(u)\,du = \int_{g^{-1}([a,b])} f(g(x))\,|g'(x)|\,dx$$



Note: $\boxed{\ell(g(I)) \approx \underbrace{|g'(x^*)|}_{\text{magnification factor.}}\, \ell(I)}$

$$\int_{g(a)}^{g(b)} f(u)\,du \approx \cancel{\sum f(x_i^*)}$$

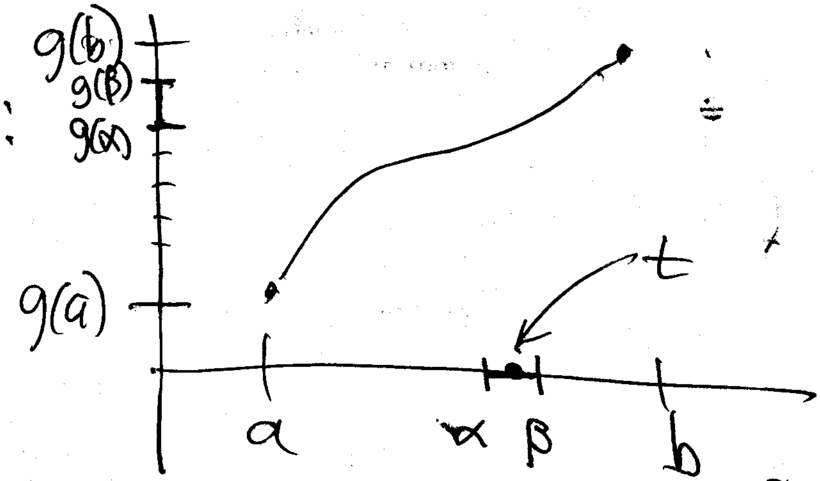
$$\sum f(u_i^*)\, \ell(g(I_i))$$

$$\approx \sum f(u_i^*)\, |g'(x_i^*)|\, \ell(I_i)$$

$$= \sum f(g(x_i^*))\, |g'(x_i^*)|\, \ell(I_i)$$

$$\approx \int_a^b f(g(x))\, |g'(x)|\, dx$$

Lemma: Let g be C^1 on $[a,b]$, $g' \neq 0$ on $[a,b]$.
Then $\ell(g([a,b])) = \int_{[a,b]} |g'(t)|\,dt$

Pf: 

Let $[\alpha,\beta] \subseteq [a,b]$ choose $t \in [\alpha,\beta]$. By Taylors formula,

$$g(x) = g(t) + g'(t)(x-t) + R(x-t)$$

where $R(x-t) = o(|x-t|)$, i.e. given $\varepsilon > 0$ there is a $\delta_0 > 0$ such that for any x, $|x-t| < \delta_0 \Rightarrow |R(x-t)| < \varepsilon |x-t|$

Letting $x=\alpha$ and $x=\beta$ we get

$$g(\alpha) = g(t) + g'(t)(\alpha - t) + R_1(\alpha - t)$$

$$g(\beta) = g(t) + g'(t)(\beta - t) + R_2(\beta - t)$$

Therefore

$$|g(\beta)-g(\alpha)| = |g'(t)(\beta - t - \alpha + t) + R_1(\alpha - t) + R_2(\beta - t)|$$

$$= |g'(t)(\beta-\alpha) - R_1(\alpha-t) + R_2(\beta-t)|$$

This implies

$$|g(\beta)-g(\alpha)| \leq |g'(t)|\,|\beta-\alpha| + (|R_1(\alpha-t)| + |R_2(\beta-t)|)$$

$$|g(\beta)-g(\alpha)| \geq |g'(t)|\,|\beta-\alpha| - (|R_1(\alpha-t)| + |R_2(\beta-t)|)$$

This implies that given $\varepsilon>0$ there is a $\delta_0>0$ such that if $|\beta-\alpha|<\delta_0$ then so is $|\alpha-t|$ and $|\beta-t|$ so that $|R_1(\alpha-t)| < \varepsilon|\alpha-t|$ and $|R_2(\beta-t)| < \varepsilon|\beta-t|$ so

$$|R_1(\alpha-t)| + |R_2(\beta-t)| < \varepsilon|\beta-\alpha|$$

Same δ_0 works for all $t\in[\alpha,\beta]$.

$\therefore$ If $|\beta-\alpha|$ is smaller than δ_0 then

$$|g(\beta)-g(\alpha)| = \ell(g([\alpha,\beta])) \leq |g'(t)|\,\ell([\alpha,\beta]) + \varepsilon\,\ell([\alpha,\beta])$$

and $|g(\beta)-g(\alpha)| \geq |g'(x)|\, \ell([\alpha\beta])$

$$- \varepsilon\, \ell([\alpha,\beta])$$

More next time...