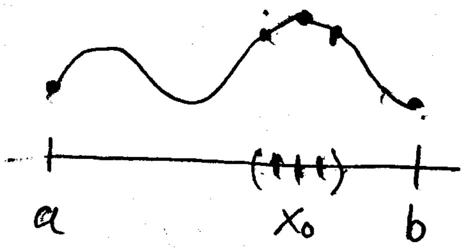


$\forall x \in (x_0-\delta, x_0+\delta)$

$f(x) \le f(x_0)$

Want $\exists\, x_1, x_2 \in (x_0-\delta, x_0+\delta)$

$f(a) = f(b)$

s.t. $x_1 \neq x_2$ $f(x_1) = f(x_2)$.

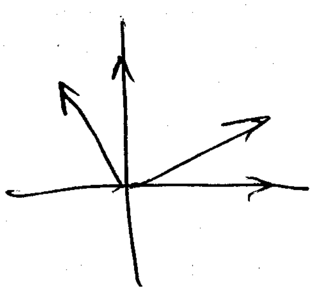
$T : E^n \to E^n$

$\overset{\mathcal{A}}{\{a_1, \dots, a_n\}}$ ~~basis~~ $\overset{\mathcal{B}}{\{b_1, \dots, b_n\}}$ bases for E^n!

$[T]_{\mathcal{A}\to\mathcal{B}} = [\beta_{ij}]$ $\quad [T]_{\mathcal{A}'\to\mathcal{B}'} = [\beta'_{ij}]$

$\vec{x} \in E^n \longrightarrow \vec{x} = \sum \alpha_i \vec{a}_i$

$T\vec{x} = \sum \alpha_i \underbrace{T(\vec{a}_i)}$



$T(\vec{a}_i) = \sum \beta_{ji} \vec{b}_j$

$= \left(\sum_i \sum_j \alpha_i \beta_{ij}\right) \vec{b}_j$

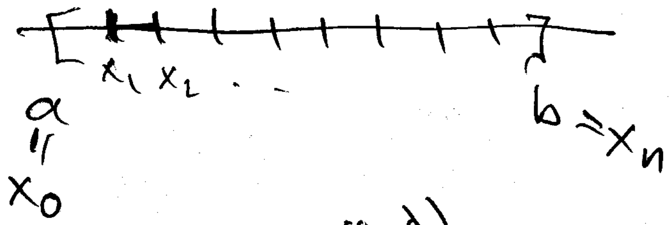
11.1 Definition of the Integral.

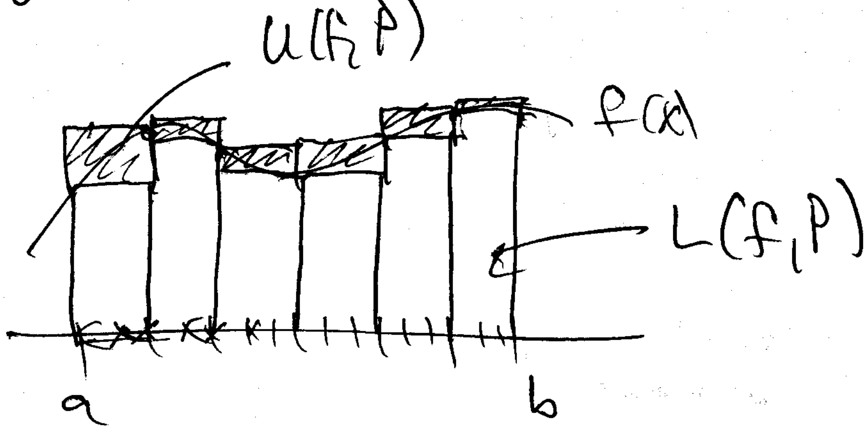
A. Integration on the line.

1. Definition 1. A *partition* P of an interval $[a,b]$ is a finite set $P = \{x_0, x_1, \dots, x_n\}$ where
$$a = x_0 < x_1 < \cdots < x_n = b.$$
Given a bounded function f defined on $[a,b]$, define
$$M_k = \sup\{f(x): \ x \in [x_{k-1}, x_k]\},$$
and
$$m_k = \inf\{ f(x): \ x \in [x_{k-1}, x_k]\}.$$

Define the *upper and lower sums* of f by
$$U(f,P) = \sum_{k=1}^{n} M_k \, (x_k - x_{k-1})$$
$$L(f,P) = \sum_{k=1}^{n} m_k \, (x_k - x_{k-1})$$
If
$$\sup_P L(f,P) = \inf_P U(f,P).$$
then $\int_a^b f(x)dx$ is this common number and f is said to be *Riemann integrable on* $[a,b]$ or $f \in \mathcal{R}[a,b]$.

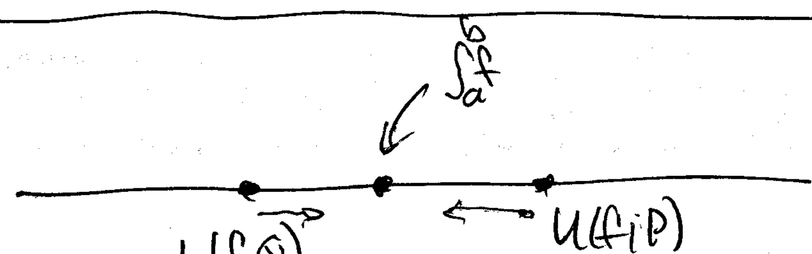
$$P = \{x_0, \dots, x_n\}$$





$$L(f,P) \le \int_a^b f \le U(f,P)$$

(I)



(i) $L(f,Q) \le U(f,P)$

(ii) P' refinement of P, i.e. $P \subseteq P'$

$$L(f,P) \le L(f,P') \qquad U(f,P) \ge U(f,P').$$

Are there functions not in $\mathcal{R}[a,b]$?

$f(x) = 1_{\mathbb{Q}}$ on $\underline{[0,1]}$ $\quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Note: For any partition P of $[a,b]$,

$$L(f,P) = 0 \qquad U(f,P) = 1$$

Problem: f is discontinuous at every point of $[0,1]$.

Also its graph cannot be covered by small rectangles.

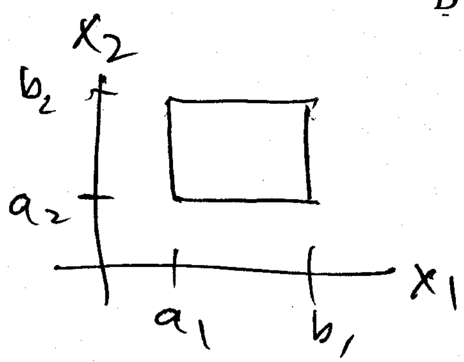
B. Integration on $\mathbb{E}^n$.

1. What replaces the interval $[a, b]$?

<u>Definition 2</u>. A *closed rectangular block* (or just a *rectangle*) $B = [\mathbb{a}, \mathbb{b}] \subseteq \mathbb{E}^n$ is defined by

$$B = [\mathbb{a}, \mathbb{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

↳ $\{(x_1, \dots, x_n) : x_i \in [a_i, b_i], \forall i\}$.

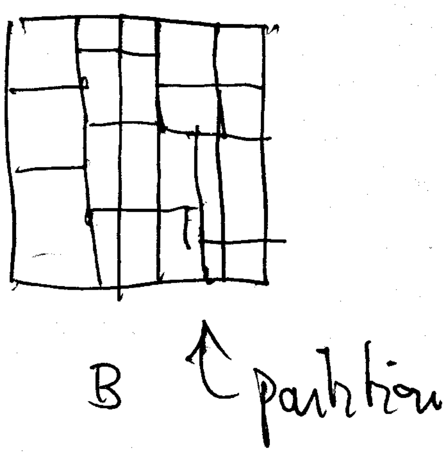


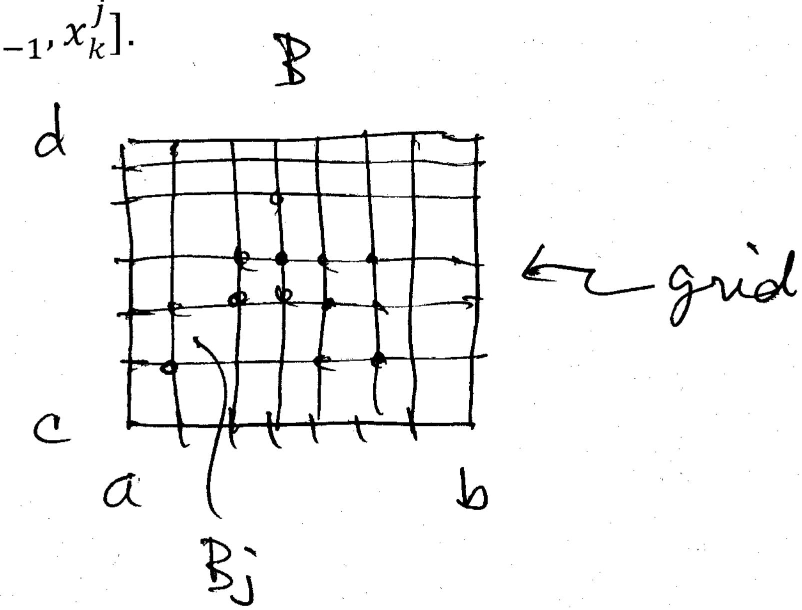
2. What replaces a partition?

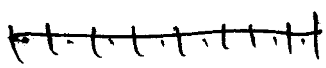
<u>Definition</u> 3. Given a rectangle $[\mathbb{a}, \mathbb{b}]$ in $\mathbb{E}^n$, a *partition* $\mathcal{P}$ of $[\mathbb{a}, \mathbb{b}]$ is the Cartesian product

$$\mathcal{P} = P_1 \times \cdots \times P_n$$

where each P_i is a partition of $[a_i, b_i]$. A *grid* is a collection of rectangles of the form $I_1 \times \cdots \times I_n$ where each I_j is a subinterval of the partition P_j, that is, $I_j = [x^j_{k-1}, x^j_k]$.







3. We also need to specify the notion of volume.

Definition 4. Let $S \subseteq \mathbb{E}^n$ and suppose that $S \subseteq B$ for some rectangle B. Let $\mathcal{P}$ be a grid on B, and define

$$V(S,\mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \cap \bar{S} \neq \emptyset}} vol(B_j);$$

$$v(S,\mathcal{P}) = \sum_{\substack{B_j \in \mathcal{P} \\ B_j \subseteq S^\circ}} vol(B_j)$$

where $\bar{S}$ is the closure and S° the interior of S.

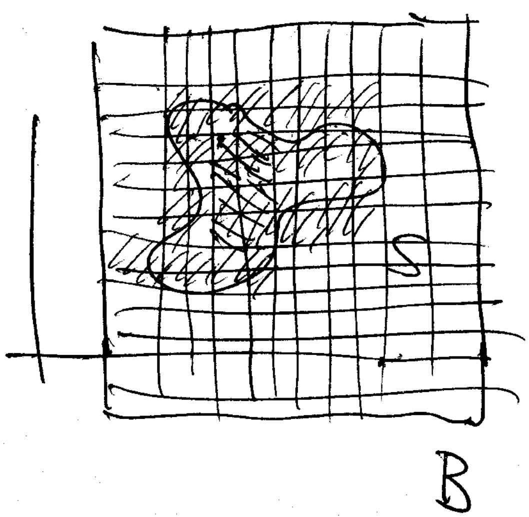
The *inner volume* of S is

$$vol(S) = \sup\{\, v(S,\mathcal{P}) : \mathcal{P} \text{ a grid on } B\}$$

and the *outer volume* is

$$Vol(S) = \inf\{\, V(S,\mathcal{P}) : \mathcal{P} \text{ a grid on } B\}$$

If $vol(S) = Vol(S)$ then S is called a *Jordan region* and the common value is the *volume* or *Jordan content* of S and is denoted $v(S)$.



$S = [0,1]^2 \cap \mathbb{Q}^2$

$= \{(x,y) : x,y \in [0,1],\ x,y \in \mathbb{Q}\}.$

$\bar{S} = [0,1]^2$

$S^\circ = \emptyset$

$v(S,P) = 0 \ \forall P \qquad V(S,P) = 1$ all P.

Remarks: (1) A ball, $\emptyset$, a rectangle are Jordan-regions

(2) $v(\cdot)$ (volume) satisfies all reasonable properties:

(i) $S_1 \subseteq S_2$, $v(S_1) \leq v(S_2)$

(ii) $v(S_1 \cup S_2) \leq v(S_1) + v(S_2)$

and $=$ if $S_1 \cap S_2 = \emptyset$

(iii) in fact if $v(S_1 \cap S_2) = 0$ then

$v(S_1 \cup S_2) = v(S_1) + v(S_2)$.

4. <u>Theorem</u>. A bounded set $S \subseteq \mathbb{E}^n$ is a Jordan region if and only if $Vol(\partial S) = 0$ where $\partial S = \bar{S} \setminus S^\circ$ is the boundary of S.

Such a set is called a *Jordan null set*.