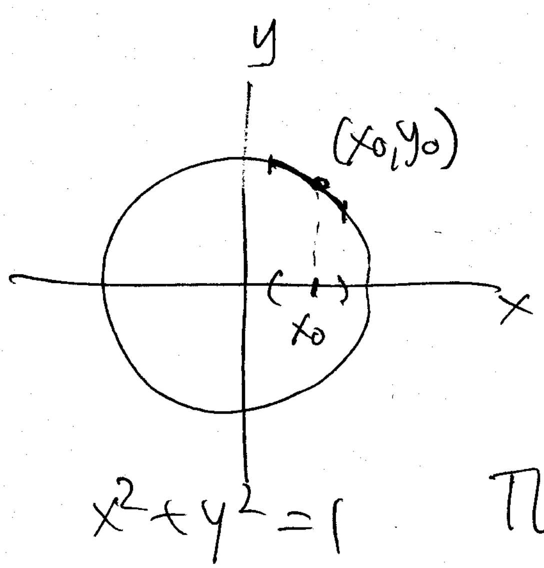
10.5 Implicit Functions.

A. Examples.

1. Consider the equation $x^2 + y^2 = 1$. Does there exist a function $g(x)$ such that for all $x \in [-1,1]$, $x^2 + g(x)^2 = 1$? In other words, can we solve this equation for y in terms of x?

   Answer: Clearly no in general, but we can say the following: Given a point (x_0, y_0) on the curve for which $\frac{dy}{dx} = -\frac{x}{y}$ is defined (that is, if $y_0 \neq 0$), there is a small open interval (a, b) containing x_0 such that such a $g(x)$ exists for $x \in (a, b)$. Also note that if $y_0 = 0$ then no such interval or function exists.



$\frac{dy}{dx} \neq 0. \Rightarrow$ can solve for y in terms of x in a small interval around x_0.

That is, $\exists\, g : (x_0-\delta, x_0+\delta) \to \mathbb{R}$ such that $x^2+g(x)^2=1$ all x in $(x_0-\delta, x_0+\delta)$.

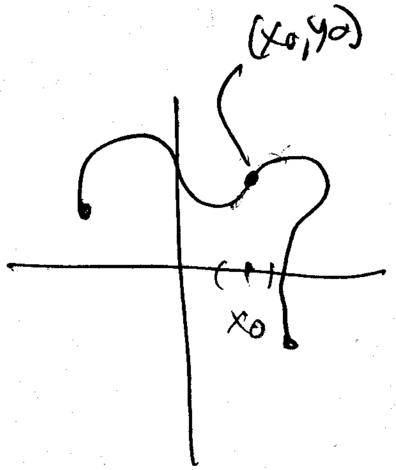
$\left(g(x) = (1-x^2)^{1/2} \right)$.

2. Suppose we have a curve in the plane given by $f(x,y) = 0$.

$$\frac{dy}{dx} = {-\frac{\partial f}{\partial x}}\Big/{\frac{\partial f}{\partial y}}$$

so the curve has a nonvertical tangent exactly when $\frac{\partial f}{\partial y} \neq 0$ and we should be able to solve for y in terms of x. But how?

3. Define $F\colon \mathbb{E}^2 \to \mathbb{E}^2$ by $F(x,y) = (x, f(x,y))$.



$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

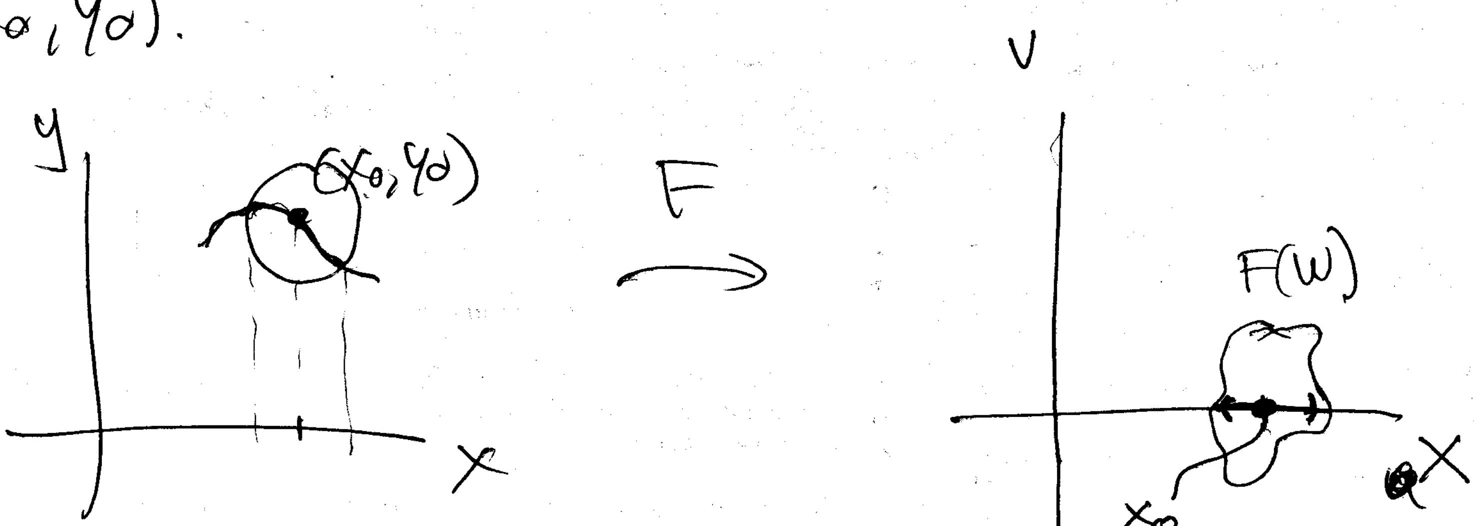
$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}. \qquad \text{If } \frac{\partial f}{\partial y} \neq 0$$

then ~~the~~ near x_0 we should be able to solve for y in terms of x.

$$F'(x,y) = \begin{bmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \quad \det F'(x,y) = \frac{\partial f}{\partial y}(x,y).$$

If $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$ and $f(x_0,y_0) = 0$.

Then ~~F exists~~ F is invertible in an open set W containing (x_0,y_0).



$$F(x_0,y_0) = (x_0, f(x_0,y_0)) = (x_0,0)$$

x-coords don't change.

Write $F^{-1}(x,v) = (G_1(x,v), G_2(x,v))$

$$F(F^{-1}(x,v)) = (x,v) = (G_1(x,v), f(G_1(x,v), G_2(x,v)))$$

~~Let v = 0~~ $\quad = (x, f(x, G_2(x,v)))$

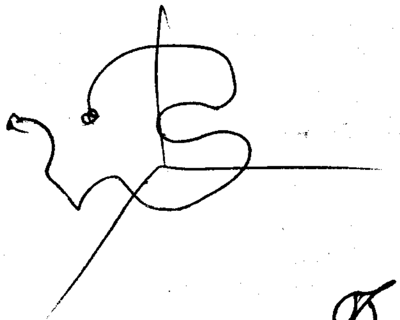
Let $v=0$: $(x,0) = (x, f(x, G_2(x,0)))$

$\therefore f(x, G_2(x,0)) = 0$ in some interval around x_0

Let $g(x) = G_2(x,0)$ we are done, i.e. $y = g(x)$.

More general approach.

Curve in $\mathbb{E}^n$: $f : \mathbb{E}^n \to \mathbb{E}^1$



$f(x_1, \ldots, x_n) = 0$ defines curve.

want to solve for x_n in terms of $x_1, \ldots, x_{n-1}$.

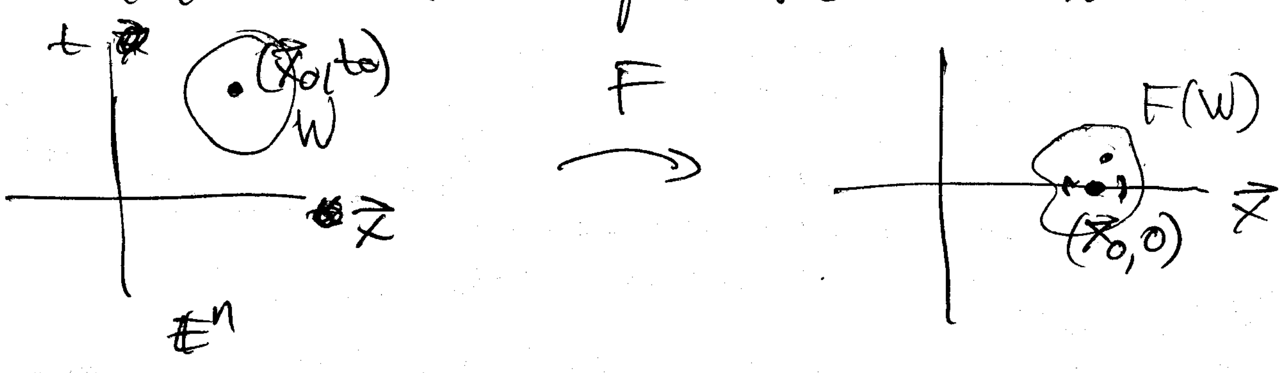
Write $(\vec{x}, t) \in \mathbb{E}^n$, $\vec{x} \in \mathbb{E}^{n-1}$, $t \in \mathbb{E}^1$

$$(\underbrace{x_1, \ldots, x_{n-1}}_{\vec{x}}, \underbrace{x_n}_{t}) \quad \text{so} \quad f(\vec{x}, t) = 0$$

<u>Idea</u>: $F : \mathbb{E}^n \to \mathbb{E}^n$ $\quad F(\vec{x}, t) = (\vec{x}, f(\vec{x}, t))$

$$F'(\vec{x}, t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_{n-1}} & \frac{\partial f}{\partial t} \end{bmatrix} ; \det F'(\vec{x}, t) = \frac{\partial f}{\partial t}(\vec{x}, t)$$

If $f(\vec{x}_0, t_0) = 0$, $\frac{\partial f}{\partial t}(\vec{x}_0, t_0) \neq 0$. Then F is invertible in an open set W cont. $(\vec{x}_0, t_0)$.



$$F(\vec{x}, t) = (\vec{x}, v).$$

$$F^{-1}(\vec{x}, v) = (G_1(x,v), \dots, G_{n-1}(x,v), G_n(x,v))$$

$$F(F^{-1}(\vec{x},v)) = (\vec{x}, v) = \cancel{(G_1(\vec{x},v), \dots, G_{n-1}(\vec{x},v), G_n(\vec{x},v))}$$

$$= (G_1(\vec{x},v), \dots, G_{n-1}(\vec{x},v), f(G_1, \dots, G_{n-1}, G_n))$$

$$= (\vec{x}, f(\vec{x}, G_n(\vec{x},v)))$$

Set $v = 0$. $(\vec{x}, 0) = (\vec{x}, f(\vec{x}, G_n(\vec{x}, 0)))$

Let $g(\vec{x}) = G_n(\vec{x}, 0)$ then

$f(\vec{x}, g(\vec{x})) = 0$ for all $\vec{x}$ near $\vec{x}_0$.

B. Theorem (10.5.1) Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^m)$ where D is an open subset of $\mathbb{E}^{n+m}$ and that for some point $(\mathbb{x}_0, \mathbb{y}_0) \in D$, $\mathbb{f}(\mathbb{x}_0, \mathbb{y}_0) = \mathbb{0}$ and

$$\frac{\partial(f_1, \dots, f_m)}{\partial(y_1, \dots, y_m)}(\mathbb{x}_0, \mathbb{y}_0) \neq 0$$

Then there is an open set $U \subseteq \mathbb{E}^n$ containing $\mathbb{x}_0$ and a function $\mathbb{g} \in C^1(U, \mathbb{E}^m)$ such that for all $\mathbb{x} \in U$, $\mathbb{f}(\mathbb{x}, \mathbb{g}(\mathbb{x})) = \mathbb{0}$.

Write: $(\vec{x}, \vec{y}) \in \mathbb{E}^{n+m}$, $\vec{x} \in \mathbb{E}^n$, $\vec{y} \in \mathbb{E}^m$

so $(\vec{x}, \vec{y}) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$

~~Define~~ Solve for $y_1, \dots, y_m$ in terms of $x_1, \dots, x_n$. We are looking for

$y_1 = g_1(x_1, \dots, x_n)$

$y_2 = g_2(x_1, \dots, x_n)$ or $g : U \subseteq \mathbb{E}^n \to \mathbb{E}^m$

$\vdots$

$y_m = g_m(x_1, \dots, x_n)$

$F(\vec{x}, \vec{y}) = (\vec{x}, f(\vec{x}, \vec{y}))$

$F : \mathbb{E}^{n+m} \to \mathbb{E}^{n+m}$.

$$F'(\vec{x},\vec{y}) = \begin{array}{c} \\ n \\ \\ \\ \\ m \\ \\ \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline \frac{\partial f_1}{\partial x_1} & \cdots & & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & & \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} & \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{array}\right]$$

$$\qquad\qquad\qquad\qquad n \qquad\qquad\qquad\qquad m$$

$$= \begin{bmatrix} I_{n\times n} & O_{n\times m} \\ \left[\frac{\partial f_i}{\partial x_j}\right]_{m\times n} & \left[\frac{\partial f_i}{\partial y_j}\right]_{m\times m} \end{bmatrix}$$

$$\det F'(\vec{x},\vec{y}) = \det\left[\frac{\partial f_i}{\partial y_j}\right]_{m\times m}(\vec{x},\vec{y}) = \frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_m)}(\vec{x},\vec{y})$$

Suppose $f(\vec{x}_0,\vec{y}_0)=\vec{0}$ and

$$\frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_m)}(\vec{x}_0,\vec{y}_0) \neq 0$$

$\therefore$ F is invertible in an open set $W \subseteq E^{n+m}$ containing $(\vec{x}_0,\vec{y}_0)$.

$$F(F^{-1}(\vec{x},\vec{v})) = (\vec{x}, f(\vec{x}, G_{n+1}(\vec{x},\vec{v}), \dots, G_{n+m}(\vec{x},\vec{v})))$$
$$= (\vec{x},\vec{v}).$$

$$\left(F^{-1}(\vec{x},\vec{v}) = (G_1(\vec{x},\vec{v}), \dots, G_n(\vec{x},\vec{v}), G_{n+1}(\vec{x},\vec{v}), \dots, G_{n+m}(\vec{x},\vec{v}))\right)$$

Set $\vec{v} = 0$. Then

$$g(\vec{x}) = \begin{bmatrix} G_{n+1}(\vec{x},0) \\ \vdots \\ G_{n+m}(\vec{x},0) \end{bmatrix} \text{ works.}$$

Remains to show that $g \in C^1(U, E^m)$ for some $U \subseteq E^n$. How exactly do you define U?