C. Inverse Function Theorem.

1. Lemma. (Open Mapping Theorem, Thm. 10.4.2). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$, then $\mathbb{f}$ is an open mapping, that is, $\mathbb{f}$ maps open subsets of D to open subsets of $\mathbb{E}^n$.

Last time we showed that if $f \in C^1(D, \mathbb{E}^n)$ $D \subseteq \mathbb{E}^n$ open the if $\det(f'(\vec{x})) \neq 0$ on D then f is locally 1-1 on D. We also know from earlier: Suppose $g: S \subseteq \mathbb{E}^n \to \mathbb{E}^m$, S compact, $g \in C(S)$. Then if g is 1-1 on S then g^{-1} is continuous on $g(S)$ (i.e. g is an open mapping)

Let $\mathcal{O} \subseteq D$ be open. We want to show $f(\mathcal{O})$ is also open. Let $\vec{y} \in f(\mathcal{O})$.

Then there is an $\vec{x} \in \mathcal{O}$ such that $\vec{y} = f(\vec{x})$ $\mathbb{E}^n$
By assumption on $f'(\vec{x})$ we know f is locally 1-1 on D. Hence for some $\varepsilon > 0$, f is 1-1 on $B(\vec{x}, \varepsilon)$. But $B(\vec{x}, \varepsilon)$ is not compact and f

need not be 1-1 on $\overline{B(\vec{x},\varepsilon)}$ which is compact.
But f is 1-1 on $\overline{B(\vec{x},\varepsilon/2)}$ which is compact.
Also, $B(\vec{x},\varepsilon/2)$ is relatively open in $\overline{B(\vec{x},\varepsilon/2)}$, hence $f(B(\vec{x},\varepsilon/2))$ is open in E^n. Since $f(\vec{x})=\vec{y}$, $\vec{y}\in f(B(\vec{x},\frac{\varepsilon}{2}))$.
Therefore there is a $\delta>0$ such that $B(\vec{y},\delta)\subseteq f(B(\vec{x},\varepsilon/2))\subseteq f(\mathcal{O})$. Hence $f(\mathcal{O})$ is open.

2. <u>Theorem</u> (10.4.3). Suppose $\mathbb{f} \in C^1(D, \mathbb{E}^n)$ where $D \subseteq \mathbb{E}^n$ is open. If $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$, and if $\mathbb{f}$ is globally one-to-one on D, then $\mathbb{f}^{-1} \in C^1(f(D), \mathbb{E}^n)$ and

$$(\mathbb{f}^{-1})'(\mathbb{f}(\mathbb{x})) = (\mathbb{f}'(x))^{-1}$$

First of all, if f^{-1} is differentiable on $f(D)$ then $f \circ f^{-1} = id$. By Chain Rule $(f\circ f^{-1})'(\vec{y}) = f'(f^{-1}(y))(f^{-1})'(\vec{y}) = id$, so for $\vec{y} \in f(D)$, $(f^{-1})'(\vec{y}) = [f'(f^{-1}(\vec{y}))]^{-1}$. Letting $f(\vec{x}) = \vec{y}$ we have $(f^{-1})'(f(\vec{x})) = f'(\vec{x})^{-1}$. Also since $\det(f'(\vec{x})) \neq 0$ on D, $[f'(\vec{x})]^{-1}$ is continuous on D. Therefore $f^{-1} \in C(f(D), E^n)$.

It remains to show that $(f^{-1})'$ exists at each $\vec{y} \in f(D)$. Let $\vec{y}_0 \in f(D)$ and let $\vec{x}_0 \in D$ satisfy $f(\vec{x}_0) = \vec{y}_0$. Note that $\vec{x}_0$ is unique since f is <u>globally</u> 1-1 on D. We need to show that

$$\|f^{-1}(\vec{y}_0+\vec{h}) - f^{-1}(\vec{y}_0) - [f'(f^{-1}(\vec{y}_0))]^{-1}\vec{h}\| \to 0 \text{ faster}$$

than $\vec{h}$. Note that

$$\|f'(f^{-1}(\vec{y}_0))^{-1}[f'(f^{-1}(\vec{y}_0))(f^{-1}(\vec{y}_0+\vec{h}) - f^{-1}(\vec{y}_0)) - \vec{h}]\|$$

Let us write $f^{-1}(\vec{y}_0) = \vec{x}_0$ for simplicity and $f^{-1}(\vec{y}_0+\vec{h}) = \vec{x}_0 + \vec{s}$ where $\vec{s} = f^{-1}(\vec{y}_0+\vec{h}) - f^{-1}(\vec{y}_0)$.

So $f'(f^{-1}(\vec{y}_0))\left[f^{-1}(\vec{y}_0+\vec{h})-f^{-1}(\vec{y}_0)\right]-\vec{h}$ becomes
$f'(\vec{x}_0)\vec{s}-\vec{h} = f'(\vec{x}_0)\vec{s}-(f(\vec{x}_0+\vec{s})-f(\vec{x}_0))$ since
$\vec{h} = (\vec{y}_0+\vec{h})-\vec{y}_0 = f(\vec{x}_0+\vec{s})-f(\vec{x}_0)$. Therefore

$$\frac{\| f'(f^{-1}(\vec{y}_0))^{-1}\left[f'(f^{-1}(\vec{y}_0))(f^{-1}(\vec{y}_0+\vec{h})-f^{-1}(\vec{y}_0))-\vec{h}\right]\|}{\|\vec{h}\|}$$

$$= \frac{\| f'(\vec{x}_0)^{-1}\left[f'(\vec{x}_0)\vec{s}-f(\vec{x}_0+\vec{s})-f(\vec{x}_0)\right]\|}{\|\vec{h}\|}$$

$$\le \| f'(\vec{x}_0)^{-1}\| \frac{\| f(\vec{x}_0+\vec{s})-f(\vec{x}_0)-f'(\vec{x}_0)\vec{s}\|}{\|\vec{h}\|}$$

It will be enough to show that

$$\lim_{\vec{h}\to 0} \frac{\| f(\vec{x}_0+\vec{s})-f(\vec{x}_0)-f'(\vec{x}_0)\vec{s}\|}{\|\vec{h}\|} = 0.$$

<u>Claim</u>: There exist constants $\alpha, r > 0$ such that if $\|\vec{s}\| = \|f^{-1}(\vec{y}_0+\vec{h})-f^{-1}(\vec{y}_0)\| < r$, then $\|\vec{h}\| \ge \alpha\|\vec{s}\| = \alpha\|f^{-1}(\vec{y}_0+\vec{h})-f^{-1}(\vec{y}_0)\|$.

<u>Remarks</u>: (1) Note that the claim can be rewritten as: If $\|\vec{s}\| < r$ then $\|f(\vec{x}_0+\vec{s})-f(\vec{x}_0)\| \ge \alpha\|\vec{s}\|$

(2) This is Theorem 10.4.1 in the book.

Assume that the claim is true and let $\varepsilon > 0$. Since $f'(\vec{x}_0)$ exists we can choose $\delta_1 > 0$ so that if $\|\vec{s}\| < \delta_1$ then $\dfrac{\|f(\vec{x}_0+\vec{s}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{s}\|}{\|\vec{s}\|} < \alpha\varepsilon$

If in addition we take $\|\vec{s}\| < r$ then since $\dfrac{\alpha}{\|\vec{h}\|} \le \dfrac{1}{\|\vec{s}\|}$,

$$\frac{\|f(\vec{x}_0+\vec{s}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{s}\|}{\|\vec{h}\|}$$

$$\le \frac{\|f(\vec{x}_0+\vec{s}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{s}\|}{\|\vec{s}\|}\left(\frac{1}{\alpha}\right) < \varepsilon$$

Since $\vec{s} = f^{-1}(\vec{y}_0+\vec{h}) - f^{-1}(\vec{y}_0)$ and f^{-1} is continuous at $\vec{y}_0$ we can choose $\delta > 0$ so that if $\|\vec{h}\| < \delta$ then $\|\vec{s}\| < \min(\delta_1, r)$. Hence if $\|\vec{h}\| < \delta$ then

$$\frac{\|f(\vec{x}_0+\vec{s}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{s}\|}{\|\vec{h}\|} < \varepsilon.$$

Finally we must prove the claim.

<u>Proof of Claim:</u> Consider the function

$L: D\times D\times\cdots\times D \subseteq E^{n^2} \to \mathcal{L}(E^n, E^n)$ given by

$$L(\vec{p}_1, \ldots, \vec{p}_n) = \left[\frac{\partial f_i}{\partial x_j}(\vec{p}_i)\right]_{i,j=1}^{n}.$$ Note that

$L(\vec{x}_0, \vec{x}_0, \ldots, \vec{x}_0) = f'(\vec{x}_0)$ is invertible and since $f \in C^1(D, E^n)$, all of its partials are continuous so L is continuous on $D\times\cdots\times D$. Since $L(\vec{x}_0, \ldots, \vec{x}_0)$ is invertible, L^{-1} exists and is continuous in some open set containing $(\vec{x}_0, \ldots, \vec{x}_0)$. Therefore there is an $r>0$ and an M such that $\|L(\vec{p}_1, \ldots, \vec{p}_n)^{-1}\|$ is bounded on $\overline{B((\vec{x}_0, \ldots, \vec{x}_0), r)}$

In other words, $\|L(\vec{p}_1, \ldots, \vec{p}_n)^{-1}\| \le M$ on $\overline{B((\vec{x}_0, \ldots, \vec{x}_0), r)}$

By the MVT, given $\vec{x} \in B(\vec{x}_0, r)$, there are vectors $\vec{c}_1, \ldots, \vec{c}_n$ also in $B(\vec{x}_0, r)$ such that

$$f(\vec{x}_0) - f(\vec{x}) = L(\vec{c}_1, \ldots, \vec{c}_n)(\vec{x}_0 - \vec{x}),$$ or

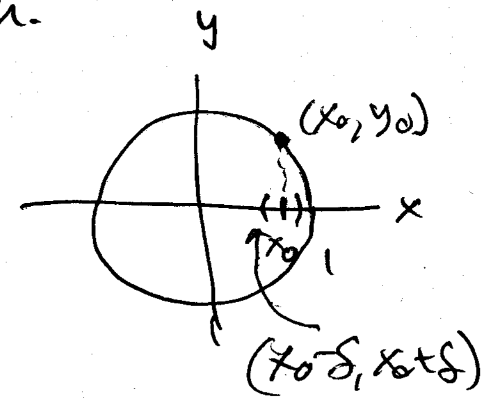
$$\vec{x}_0 - \vec{x} = L(\vec{c}_1, \ldots, \vec{c}_n)^{-1}(f(\vec{x}_0) - f(\vec{x})).$$ Hence

$$\|\vec{x}_0 - \vec{x}\| \le \|L(\vec{c}_1, \ldots, \vec{c}_n)^{-1}\| \, \|f(\vec{x}_0) - f(\vec{x})\| \le M\|f(\vec{x}_0) - f(\vec{x})\|$$

Letting $\alpha = \frac{1}{M}$ gives the result.

Implicit Function Theorem.

<u>Example</u>: $x^2+y^2=1$

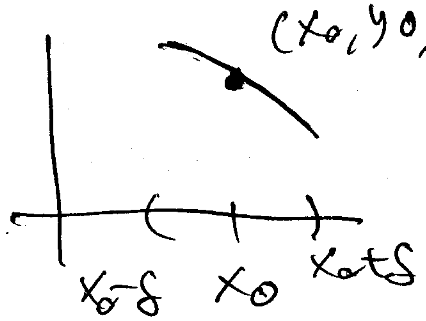


Can we solve for y in terms of x? $y^2=1-x^2$

$$y=\pm\sqrt{1-x^2}\ ?$$

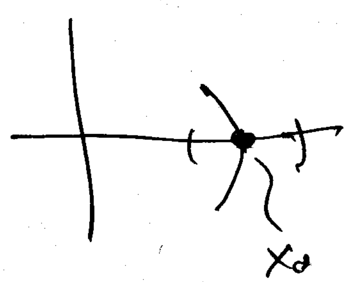
To understand this: Given (x_0, y_0) on curve we can find an open interval containing x_0 such that on this interval there is a function $g:(x_0-\delta, x_0+\delta)\to\mathbb{R}$ such that $x^2+g(x)^2=1$, i.e. $y=g(x)$ satisfies $x^2+y^2=1$.

Note that solution is valid only for x in the interval.



This works for all $x_0\in(-1,1)$.

If $x_0=1$ then there is no such $g(x)$. Why?



Note that on <u>no interval</u> containing x_0 can I find $g(x)$

Derivative:

(a) $x^2+y^2=1$. Find slope of tangent line

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$$

$$2x+2y\frac{dy}{dx}=0 \longrightarrow \frac{dy}{dx}=\frac{-x}{y}$$

vertical tangent
when $y=0$.

When tangent non-vertical, $g(x)$ exists.

(b) Suppose we have a curve in $\mathbb{R}^2$ written as $f(x,y)=0$. If we could solve for y we would have:

$$\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}(0)$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$$

Non-vertical tangent when $\frac{\partial f}{\partial y}\neq 0$.

We expect to solve for y if $\frac{\partial f}{\partial y}\neq 0$. How?

Use inverse functions:

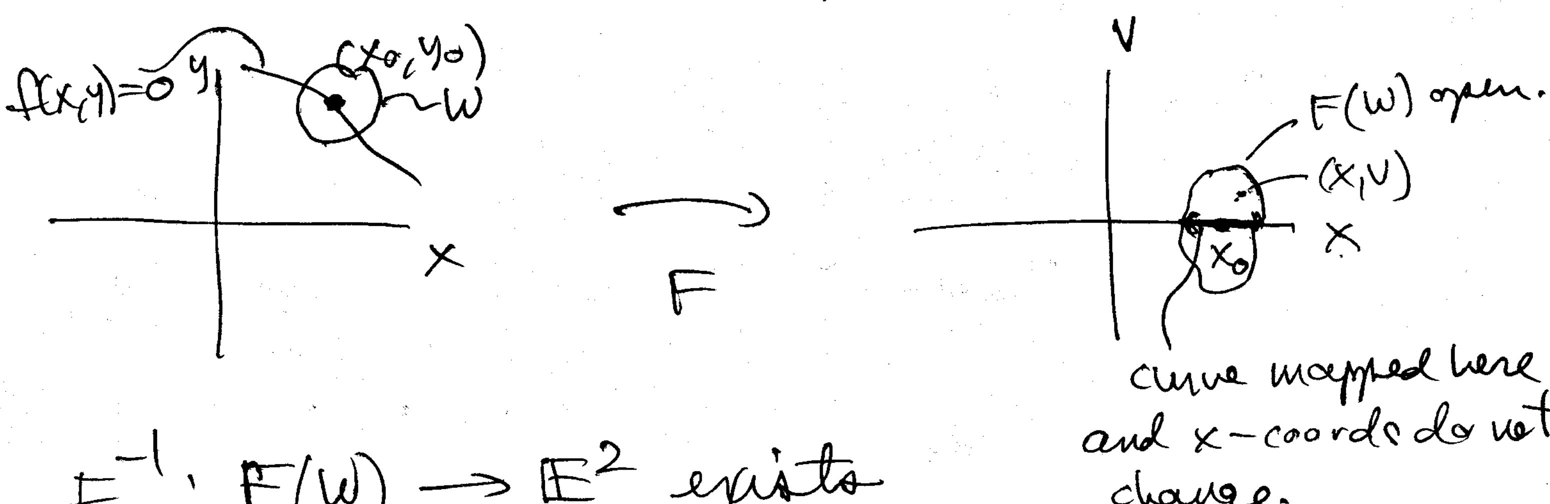
Define $F: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = (x, f(x,y)).$$

$$F'(x,y) = \begin{bmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\det(F'(x,y)) = \frac{\partial f}{\partial y} \neq 0.$$

If $f(x_0,y_0) = 0$, i.e. (x_0,y_0) is on the curve and $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$ then F is locally 1-1, i.e. there is a ball W around (x_0,y_0) such that $F^{-1} \in C^1(W, E^2)$



$F^{-1}: F(W) \to E^2$ exists

$$F^{-1}(x,v) = (G_1(x,v), G_2(x,v))$$

$$F(F^{-1}(x,v)) = (x,v) = F(G_1(x,v), G_2(x,v))$$

$$= (G_1(x,v), f(G_1(x,v), G_2(x,v)))$$

So $G_1(x,v) = x$ and

$v = f(x, G_2(x,v))$, for all $(x,v) \in F(W)$.

Setting $v=0$ gives $f(x, G_2(x,0)) = 0$ for all x such that $(x,0) \in F(W)$, i.e. in some open interval around x_0. Letting $g(x) = G_2(x,0)$, g is C^1 on the interval around x_0.