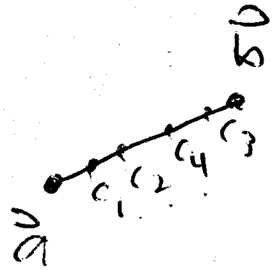
8. <u>Theorem</u>. (MVT 2) Under the hypotheses of the previous theorem, there exist vectors $\mathbb{c}_1, \mathbb{c}_2, \ldots, \mathbb{c}_m \in V \subseteq \mathbb{E}^n$ such that

$$\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a}) = \left[\frac{\partial f_i}{\partial x_j}(\mathbb{c}_j)\right](\mathbb{b} - \mathbb{a})$$



<u>Know</u>: $f : \mathbb{E}^n \to \mathbb{R}$

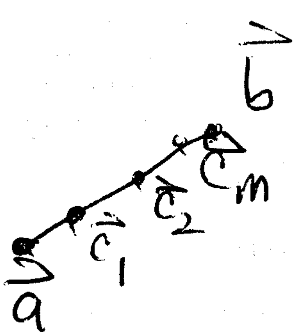
$$f'(\vec{x}) = \nabla f(\vec{x}) = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$$

MVT: $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$

where $\vec{c}$ on line joining $\vec{a}$ and $\vec{b}$.

In general $f : D \subseteq \mathbb{E}^n \to \mathbb{E}^m$

$$f(\vec{b}) - f(\vec{a}) = \underbrace{\left[\frac{\partial f_i}{\partial x_j}(\vec{c}_j)\right]}_{m \times n} \underbrace{(\vec{b} - \vec{a})}_{\substack{\mathbb{E}^n \\ n \times 1}}$$

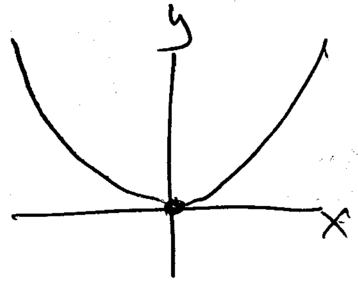


Another way: Given $\vec{a}, \vec{b}$, $\exists\, T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ such that $f(\vec{b}) - f(\vec{a}) = T(\vec{b} - \vec{a})$

10.4 Inverse Functions.

A. Local Invertibility.

1. Example. Let $f(x) = x^2$. For any $x_0 \in (0, \infty)$ there is an open interval around x_0 such that f restricted to that interval is injective. Similarly for any $x_0 \in (-\infty, 0)$. However f is not injective on any open interval containing 0.



f(x)

No $f^{-1}(x)$ exists ~~for~~ on $\mathbb{R}$

Restrict to $(0,\infty)$ or $(-\infty,0)$ and f^{-1} exists in each case.

$(0,\infty)$: $f^{-1}(x) = \sqrt{x}$

$(-\infty,0)$: $f^{-1}(x) = -\sqrt{x}$

Q: Given $x_0 \in \mathbb{R}$, is there an open interval containing x_0 on which f is invertible?

A. If $x_0 \in (0,\infty)$ then yes. If $x_0 \in (-\infty,0)$ then yes. If $x_0 = 0$, then there is no open interval containing x_0 on which f is invertible.

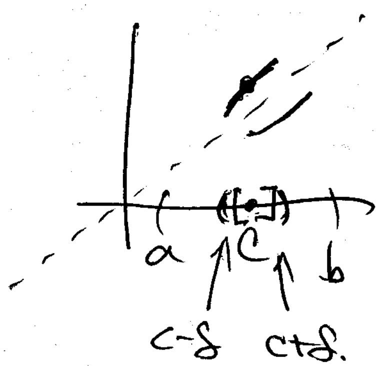
Note that $f'(0) = 0$.

2. <u>Remark</u>. Suppose $f: (a, b) \to \mathbb{R}$ is C^1 on (a, b) and that for some $c \in (a, b)$, $f'(c) \neq 0$. Then the following conclusions hold:

a. $f'(x) \neq 0$ on some interval $(c - \delta, c + \delta)$.

b. $f(x)$ is monotone on this interval, hence invertible.

c. f^{-1} is also C^1 on some open interval around $f(c)$, and $(f^{-1})'(y) = 1/f'(f^{-1}(y))$ for y in this interval.



(c) We had a theorem, the <u>Open Mapping</u> Theorem said:

~~If f^{-1} is continuous then f is open mapping~~

$f: D \to E^m$ and f 1-1 then

compact

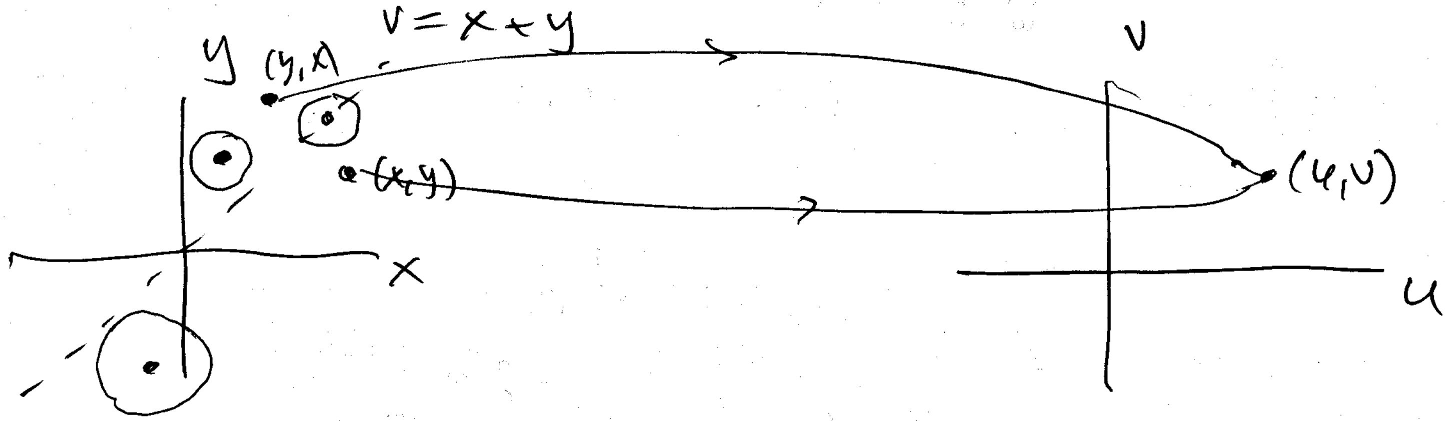
f is an <u>open mapping</u> (i.e. f^{-1} continuous).

3. <u>Example</u>. $f:\mathbb{E}^2 \to \mathbb{E}^2$ given by

$$f(x,y) = (x^2+y^2, x+y)$$

$u = x^2+y^2$

$v = x+y$



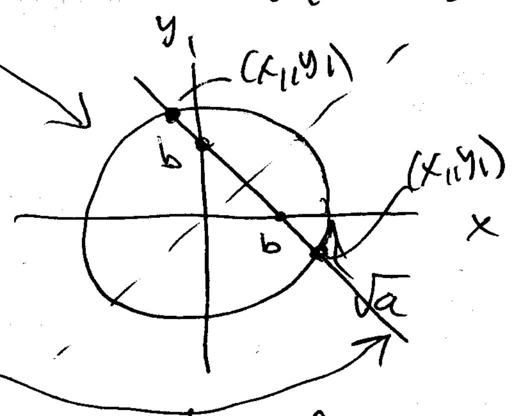
Is f one-~~to~~-one? $\left(\begin{array}{l}\text{If } f(x_1,y_1) = f(x_2,y_2) \text{ then} \\ (x_1,y_1) = (x_2,y_2)\end{array}\right)$

No. $f(x,y) = f(y,x)$ for every $(x,y) \in \mathbb{E}^2$.

$f(x_1,y_1) = f(x_2,y_2) \longrightarrow \begin{array}{l} x_1^2+y_1^2 = x_2^2+y_2^2 \\ x_1+y_1 = x_2+y_2 \end{array}$

Say $x_1^2+y_1^2 = a$

$x_1+y_1 = b.$



2 solutions or no solution

If $x \le y$ then we can solve for x,y uniquely from $x^2+y^2=a$ and $x+y=b$. So f is 1–1 on $D = \{(x,y) : x \le y\}$ and also on $D' = \{(x,y) : x \ge y\}$.

Conclusion: Given (x_0, y_0): If $x_0 < y_0$ then in a small ball centered at (x_0, y_0), f is invertible (i.e. 1-1). Similarly if $x_0 > y_0$. But if $x_0 = y_0$ then there is no open set U containing (x_0, y_0) where f is invertible.

What is f'?
$$f'(x,y) = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$$

$\det(f'(x,y)) = 2x - 2y = 2(x-y)$.

$\det f'(x,y) = 0$ ~~pr~~ precisely when $x = y$.

4. Definition. (Local invertibility) A function $\mathbb{f}\colon \mathbb{E}^n \to \mathbb{E}^n$ is *locally one-to-one* in an open set V if for every $\mathbb{x}_0 \in V$, there is an $\epsilon > 0$ such that $\mathbb{f}$ restricted to $B(\mathbb{x}_0, \epsilon)$ is one-to-one. If $\mathbb{f}$ is one-to-one on a set E then we say $\mathbb{f}$ is *globally one-to-one* on E.

5. Example. Let $f(x,y) = (x\ \cos y, x\ \sin y)$ be defined on the open set $V = \{(x,y) : x > 0\}$. Then f is locally $1-1$ on V but not globally $1-1$ on V.

If $f(x_1,y_1) = f(x_2,y_2)$ then $x_1 \cos y_1 = x_2 \cos y_2$

$x_1 \sin y_1 = x_2 \sin y_2$

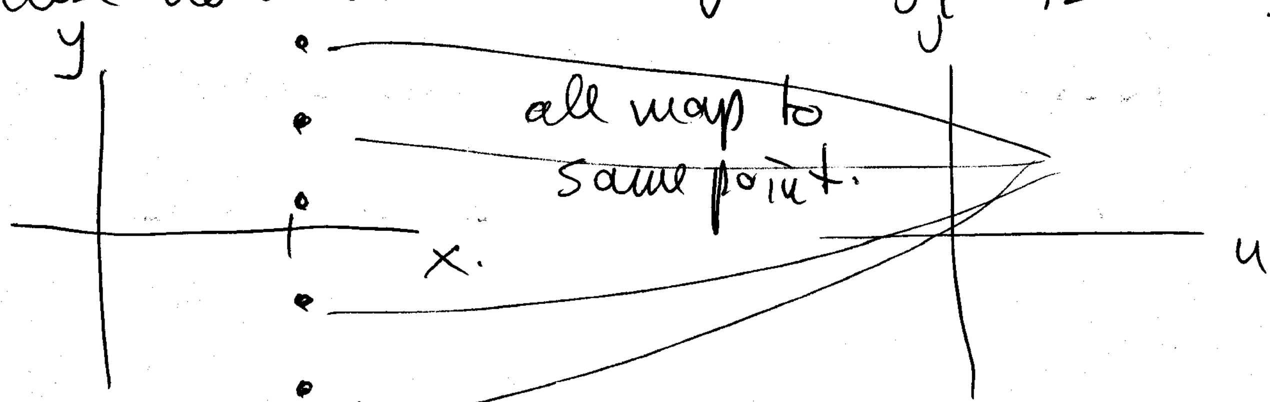
$\rightarrow x_1^2 \cos^2 y_1 + x_1^2 \sin^2 y_1 = x_2^2 \cos^2 y_2 + x_2^2 \sin^2 y_2$

$\therefore x_1^2 = x_2^2$

But if I am restricted to $V = \{(x,y) : x > 0\}$ then $x_1 = x_2$. We are left with

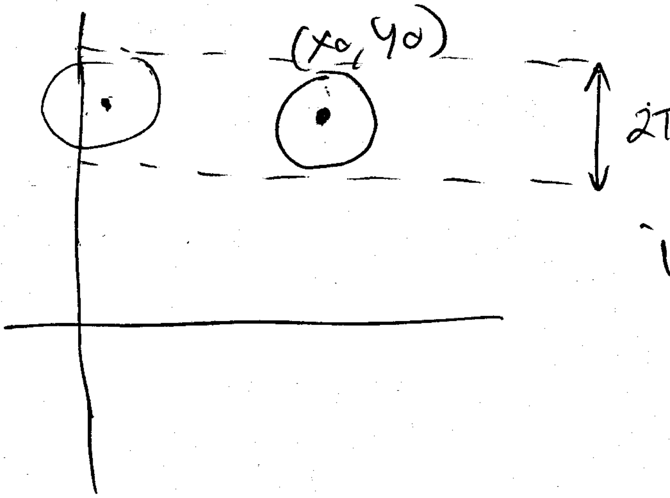
$\cos y_1 = \cos y_2 \qquad \sin y_1 = \sin y_2$

These hold if and only if $y_1 = y_2 + 2n\pi$, $n \in \mathbb{Z}$.

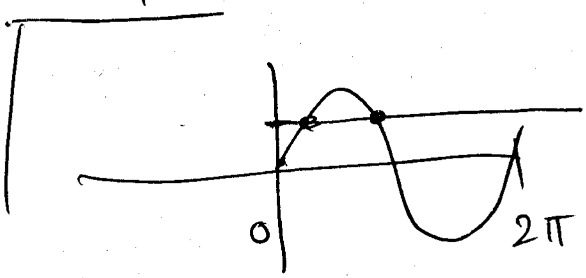


So f <u>not</u> globally 1-1. Pick $(x_0, y_0) \in V$ (so $x_0 > 0$). Choose $\varepsilon > 0$ so that

$x_0 - \varepsilon > 0$ (so I stay in V)



and $\varepsilon < \pi$ then f is 1-1 on $B((x_0, y_0), \varepsilon)$



$$f'(x,y) = \begin{bmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{bmatrix}$$

$$\det f'(x,y) = x \cos^2 y + x \sin^2 y = x$$

so $\det f'(x,y) \neq 0$ for all $(x,y) \in V$.

B. The Jacobian.

1. Definition. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$ is in $C^1(D, \mathbb{E}^n)$. Then the *Jacobian* of $\mathbb{f}$ at $\mathbb{x} \in D$ is given by $\det(\mathbb{f}'(\mathbb{x}))$.

2. Theorem. Suppose that $\mathbb{f}: D \subseteq \mathbb{E}^n \to \mathbb{E}^n$, D an open subset of $\mathbb{E}^n$, is in $C^1(D, \mathbb{E}^n)$, and suppose that $\det(\mathbb{f}'(\mathbb{x})) \neq 0$ for all $\mathbb{x} \in D$. Then $\mathbb{f}$ is locally one-to-one in D.

Pf: We want to show: Given $\vec{x}_0 \in D$ there is an $\varepsilon > 0$ such that f restricted to $B(\vec{x}_0, \varepsilon)$ is 1-1. Let $\vec{x}_0 \in D$. First, since D is open, there is an $\varepsilon_1 > 0$ such that $B(\vec{x}_0, \varepsilon_1) \subseteq D$.

[Idea: (MVT) Show $f(\vec{x}_1) = f(\vec{x}_2)$ then $\vec{x}_1 = \vec{x}_2$

$f(\vec{x}_1) - f(\vec{x}_2) = \vec{0} = T(\vec{x}_1 - \vec{x}_2)$. If T is invertible then $f(\vec{x}_1) = f(\vec{x}_2) \Rightarrow T(\vec{x}_1 - \vec{x}_2) = 0 \Rightarrow \vec{x}_1 = \vec{x}_2$]

Define the function $h: \mathbb{E}^{m\times n = n^2} \to \mathbb{E}^1 = \mathbb{R}$ by.

$$h(\vec{P}_1, \vec{P}_2, \cdots, \vec{P}_m) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{P}_1) & \frac{\partial f_1}{\partial x_2}(\vec{P}_1) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{P}_1) \\ \frac{\partial f_2}{\partial x_1}(\vec{P}_2) & \cdots & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{P}_2) \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{P}_m) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{P}_m) \end{bmatrix}$$

$(\vec{P}_i \in \mathbb{R}^n)$

$(m = n)$

Since $f \in C^1(D, E^n)$, all its partials are continuous and since the det of a matrix is just products and sums of the entries, h is continuous on D.

Also by assumption, $h(\vec{x}_0, \vec{x}_0, \vec{x}_0, \ldots, \vec{x}_0) \neq 0$

Therefore there is an $\varepsilon_2 > 0$ such that if $\| \vec{P}_i - \vec{x}_0 \| < \varepsilon_2$ for $i = 1, 2, \ldots, n$ then

$h(\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_n) \neq 0$. Now let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

If $\vec{x}_1, \vec{x}_2 \in B(\vec{x}_0, \varepsilon)$ then so is the line segment joining them. Hence by MVT there is a $T \in \mathcal{L}(E^n, E^n)$ such that

$$f(\vec{x}_2) - f(\vec{x}_1) = T(\vec{x}_2 - \vec{x}_1), \quad T = \left[\frac{\partial f_i}{\partial x_j}(\vec{P}_i) \right]_{i,j=1}^{n}$$

Since $h(\vec{P}_1, \ldots, \vec{P}_n) \neq 0$, $\det T \neq 0$, so $\vec{x}_1 = \vec{x}_2$ and f is one-to-one.