$$\frac{f(x+h)-f(x)}{h} \to f'(x)$$

$$\frac{f(x+h)-f(x)-\boxed{f'(x)}h}{h} \to 0$$

$f(x+h)-f(x)-mh \to 0$ for any m

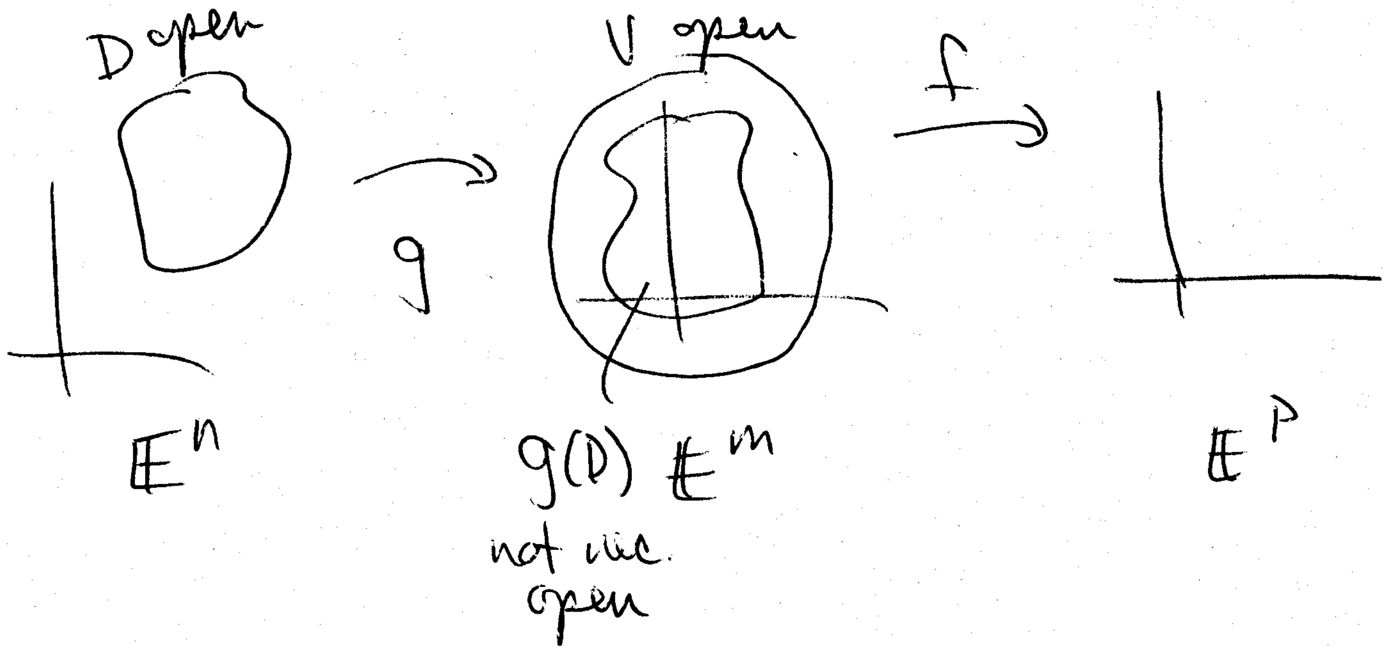
$\frac{f(x+h)-f(x)-mh}{h} \not\to 0$ unless $m=f'(x)$

B. The Chain Rule.

1. Theorem. Suppose that $\mathbb{g}: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ and $\mathbb{f}: V \subseteq \mathbb{E}^m \to \mathbb{E}^p$, where D is an open subset of $\mathbb{E}^n$ and V is an open subset of $\mathbb{E}^m$ such that $\mathbb{g}(D) \subseteq V$, and that $\mathbb{g}'(\mathbb{x}_0)$ and $\mathbb{f}'(\mathbb{g}(\mathbb{x}_0))$ both exist at $\mathbb{x}_0 \in D$. Then

$$\underset{p \times n}{(\mathbb{f} \circ \mathbb{g})'(\mathbb{x}_0)} = \underset{p \times m}{\mathbb{f}'(\mathbb{g}(\mathbb{x}_0))}\ \underset{m \times n}{\mathbb{g}'(\mathbb{x}_0)}$$

2. Remark. How do we interpret this theorem in terms of linear transformations?



$f \circ g : \mathbb{E}^n \longrightarrow \mathbb{E}^p \qquad [(f \circ g)'(\vec{x}_0)]_{p \times n}$

$f : \mathbb{E}^m \longrightarrow \mathbb{E}^p \qquad [f'(g(\vec{x}_0))]_{p \times m}$

$g : \mathbb{E}^n \longrightarrow \mathbb{E}^m \qquad [g'(\vec{x}_0)]_{m \times n}$

3. Proof of Theorem.

Need to show $f\circ g(\vec{x}_0+\vec{h}) - f\circ g(\vec{x}_0) - f'(g(\vec{x}_0))g'(\vec{x}_0)\vec{h}$ $\to 0$ faster than $\vec{h} \in \mathbb{E}^n$. Note that

$\cancel{f(\ldots)}$

$$f(g(\vec{x}_0+\vec{h})) - f(g(\vec{x}_0)) - f'(g(\vec{x}_0))g'(\vec{x}_0)\vec{h}$$

$$= \left[f(g(\vec{x}_0+\vec{h})) - f(g(\vec{x}_0)) - f'(g(\vec{x}_0))\left(g(\vec{x}_0+\vec{h}) - g(\vec{x}_0)\right)\right]$$

$$+ \left[f'(g(\vec{x}_0))\left(g(\vec{x}_0+\vec{h}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{h}\right)\right]$$

$$= (1) + (2)$$

Look at (2) first

$$\frac{\|(2)\|}{\|\vec{h}\|} = \frac{\|f'(g(\vec{x}_0))\left(g(\vec{x}_0+\vec{h}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{h}\right)\|_{\mathbb{E}^p}}{\|h\|_{\mathbb{E}^n}}$$

$$\le \|f'(g(\vec{x}_0))\|_{\mathcal{L}(\mathbb{E}^m,\mathbb{E}^p)} \frac{\|g(\vec{x}_0+\vec{h}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{h}\|_{\mathbb{E}^m}}{\|h\|_{\mathbb{E}^n}}$$

$\to 0$ as $\|\vec{h}\| \to 0$.

Look at ①. Let $\varepsilon > 0$. Need to find $\delta > 0$ such that if $\|\vec{h}\| < \delta$ then $\|①\| \le \varepsilon \|\vec{h}\|$. We know that

$$\lim_{\vec{s} \to 0} \frac{\|g(\vec{x}_0 + \vec{s}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{s}\|}{\|\vec{s}\|} = 0$$

So ~~taking~~ there is an $\eta > 0$ such that if $\|\vec{s}\| < \eta$ then $\|g(\vec{x}_0 + \vec{s}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{s}\| < \|\vec{s}\|$

(just use defn of limit with "$\varepsilon = 1$")

Therefore as long as $\|\vec{h}\| < \eta$ then

$$\|\vec{g}(\vec{x}_0 + \vec{h}) - g(\vec{x}_0)\|$$

$$\le \|g(\vec{x}_0 + \vec{h}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{h}\| + \underbrace{\|g'(\vec{x}_0)\vec{h}\|}_{\le \|g'(\vec{x}_0)\| \, \|\vec{h}\|}$$

$$< (1 + \|g'(\vec{x}_0)\|) \|\vec{h}\|.$$

$$\underbrace{g'(\vec{x}_0)}_{\mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)}$$

For $\varepsilon > 0$ above choose $\delta > 0$ so that if $\|\vec{s}\| < \delta$ then

$$\|\boxed{f(g(\vec{x}_0) + \vec{s}) - f(g(\vec{x}_0))} - f'(g(\vec{x}_0))\vec{s}\| < \frac{\varepsilon}{1 + \|g'(x_0)\|} \|\vec{s}\|$$

Now if $\|\vec{h}\| < \min\left(\eta, \frac{\delta}{1 + \|g'(x_0)\|}\right)$

Then $\|g(\vec{x}_0+\vec{h})-g(\vec{x}_0)\| < (1+\|g'(\vec{x}_0)\|)\|\vec{h}\| < \delta.$

So letting $\vec{s} = g(\vec{x}_0+\vec{h})-g(\vec{x}_0)$ we have

$$\|f(g(\vec{x}_0+h)) - f(g(\vec{x}_0)) - f'(g(x_0))(g(\vec{x}_0+\vec{h})-g(\vec{x}_0))\|$$

$$= \|f(g(\vec{x}_0)+\vec{s}) - f(g(\vec{x}_0)) - f'(g(\vec{x}_0))\vec{s}\|$$

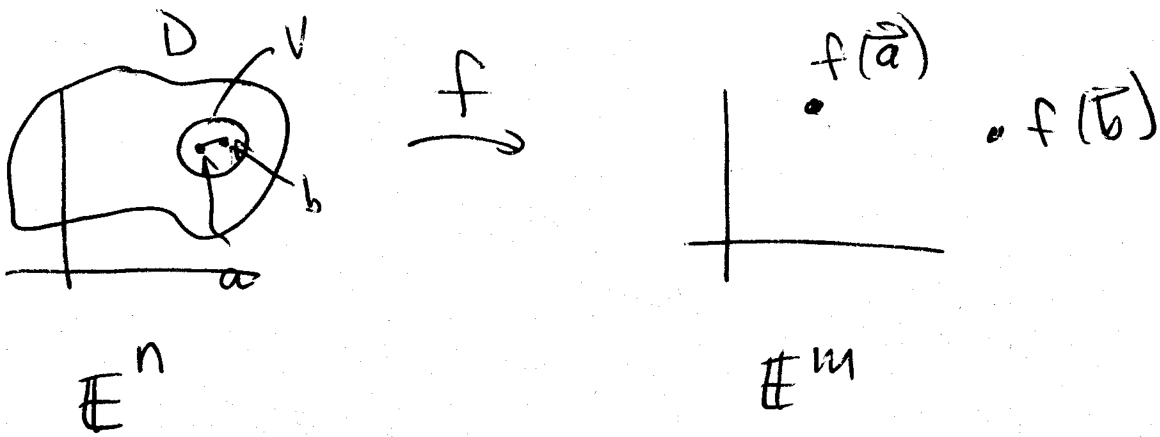
$$< \frac{\varepsilon}{1+\|g'(x_0)\|}\|\vec{s}\| = \frac{\varepsilon}{1+\|g'(x_0)\|}\|g(\vec{x}_0+\vec{h})-g(\vec{x}_0)\| < \varepsilon$$

C. The Mean Value Theorem.

1. Theorem. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there is a $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

2. Remark. A natural generalization to functions $\mathbb{f}\colon D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ might be: Suppose that $\mathbb{f}\colon V \to \mathbb{E}^m$ where V is a ball in $\mathbb{E}^n$. Then given $\mathbb{a}, \mathbb{b} \in V$ there is a $\mathbb{c}$ on the line segment joining $\mathbb{a}$ and $\mathbb{b}$ such that $\underbrace{\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a})}_{\in \mathbb{E}^m} = \underbrace{\mathbb{f}'(\mathbb{c})}_{m\times n}\underbrace{(\mathbb{b} - \mathbb{a})}_{\mathbb{E}^n}$.

3. Note first of all that the dimensions of the matrices work out, but the theorem does not hold.



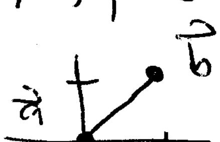
e.g. $f(t) = (\sin t, \cos t)$

$f(2\pi) = f(0) \qquad f(2\pi) - f(0) = \vec{0}$

but no $c \in (0, 2\pi)$ with $f'(c) = \vec{0}$.

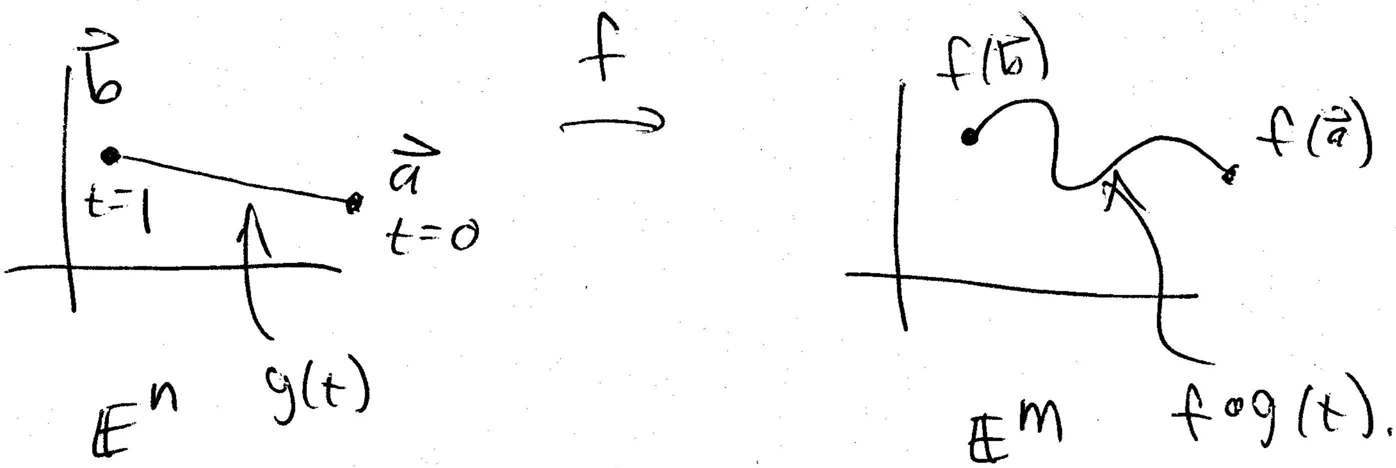
e.g. $f(x,y) = x(y-1) \qquad \vec{a} = (0,0),\ \vec{b} = (1,1)$

Then $f(\vec{b}) - f(\vec{a}) = 0$. But



$f'(x) = \nabla f = (y-1, x) \neq \vec{0}$ on line segment.

4. For $\mathbb{f}$ as above, consider the function $\mathbb{g}: \mathbb{R} \to \mathbb{E}^m$ given by $\mathbb{g}(t) = t\mathbb{b} + (1-t)\mathbb{a}$. Then look at the function $\mathbb{f} \circ \mathbb{g}: \mathbb{R} \to \mathbb{E}^m$. What can we say in this case?



So $f \circ g(t) : [0,1] \to \mathbb{E}^m$

Does MVT work here? $\exists ?\ t_0 \in (0,1)$

s.t. $\underbrace{f(\vec{b}) - f(\vec{a})}_{\mathbb{E}^m} = \underbrace{(f \circ g)'(t_0)}_{m \times 1} \cdot \underbrace{(\vec{b} - \vec{a})}_{1 \times n}$ (?)

But this does not work by previous example.

Idea: Look at each component of $f \circ g$, i.e. $(f \circ g)_j : \mathbb{E}^1 \to \mathbb{E}^1$ so ordinary MVT holds.

Since $(f\circ g)_j = \vec{e}_j \cdot (f\circ g)$ we can more generally ask:

Given $\vec{u} \in \mathbb{E}^m$ does MVT hold for $\vec{u}\cdot(f\circ g)$?

If $\vec{a}, \vec{b} \in V$ then line seg joining $\vec{a}$ to $\vec{b}$ is in V.

not convex.

5. <u>Theorem</u>. (MVT 1) Let $V \subseteq \mathbb{E}^n$ be open and convex, and let $\mathbb{f}: V \to \mathbb{E}^m$ be differentiable on V. Let $\mathbb{a}$, $\mathbb{b} \in V$ and let $\mathbb{u} \in \mathbb{E}^m$ be an arbitrary vector. Then there is a $\mathbb{c}$ on the line segment joining $\mathbb{a}$ and $\mathbb{b}$ such that

$$\mathbb{u} \cdot \big(\mathbb{f}(\mathbb{b}) - \mathbb{f}(\mathbb{a})\big) = \mathbb{u} \cdot (\underbrace{\mathbb{f}'(\mathbb{c})}_{m \times n}(\mathbb{b} - \mathbb{a}))$$

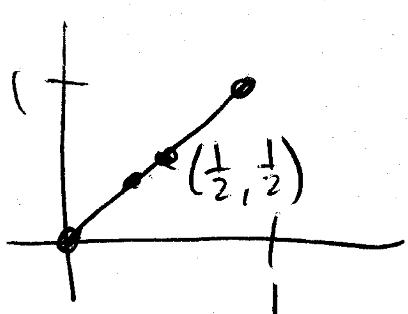
6. <u>Example</u>. Let $f(x, y) = x(y - 1)$. Then $f(1,1) - f(0,0) = 0$, and $\nabla f(x, y)$ does not vanish on the line segment joining $(0,0)$ and $(1,1)$.

$$f(1,1) - f(0,0) = \vec{0}$$

$$u\,\big(f(1,1) - f(0,0)\big) = 0$$

$$f'(\vec{c}) = \nabla f(c_1, c_2) = [\cancel{y-1, x}] = [c_2 - 1, c_1]$$

Can I find (c_1, c_2) on segment joining $(0,0)$ to $(1,1)$ such that



$$\nabla f(c_1, c_2) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$[c_2 - 1, c_1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_2 + c_1 - 1 = 0$$

Normally $\vec{c}$ will depend on $\vec{u}$.

$$\nabla f(\tfrac{1}{2}, \tfrac{1}{2}) \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b} - \vec{a}} = 0$$

7. Proof of MVT 1.

Let $g(t) = t\vec{b} + (1-t)\vec{a}$ for $t \in [0,1]$. Then $(f \circ g) : [0,1] \to E^m$ and since $t\vec{b} + (1-t)\vec{a}$ is in V for all t, $f \circ g$ is continuous on $[0,1]$. Also $f \circ g$ is differentiable on $(0,1)$ with

$$(f \circ g)'(t) = \underbrace{f'(g(t))}_{m \times n}\underbrace{g'(t)}_{n \times 1} = f'(g(t))(\vec{b} - \vec{a})$$

Now let $\underline{F(t) = \vec{u} \cdot (f \circ g)(t)}$. Then F is cont on $[0,1]$ and differentiable on $(0,1)$ with $F'(t) = \vec{u} \cdot (f \circ g)'(t) = \vec{u} \cdot [f'(g(t))(\vec{b} - \vec{a})]$

By the MVT there is a $t_0 \in (0,1)$ such that $F(1) - F(0) = F'(t_0)(1-0) = F'(t_0)$. That is

$$\vec{u} \cdot (f(\vec{b}) - f(\vec{a})) = \vec{u} \cdot [f'(g(t_0))(\vec{b} - \vec{a})]$$

$$= \vec{u} \cdot (f'(\vec{c})(\vec{b} - \vec{a})) \quad \underline{\underline{\vec{c} = g(t_0)}}.$$