$T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$

$$\underline{\|T\|} = \sup_{\vec{x} \neq 0} \frac{\|Tx\|}{\|x\|} \longrightarrow \text{smallest } K \text{ s.t. } \forall \vec{x} \in \mathbb{E}^n \quad \|Tx\|_{\mathbb{E}^m} \leq K \|x\|_{\mathbb{E}^n}$$

Know if T represented by matrix $[T] = (t_{ij})$

then $\|T\| \leq \boxed{\left(\sum_{i,j} t_{ij}^2\right)^{1/2}}$

Conjecture: $\|T\| \neq$ smallest row sum or column sum

$$[T] = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \leftarrow \|T\| = 2$$

$\mathcal{L}(\mathbb{E}^n, \mathbb{E}^m) \sim \mathbb{E}^{n \cdot m}$

$L : T \longmapsto$ vectorized version of $[T]_{B \to C}$

$\exists\, c, C > 0$ s.t. $\forall T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$

$$c\|T\| \leq \|L(T)\|_{\mathbb{E}^{nm}} \leq C\|T\|$$

Lets assume without loss that $f: E^n \to \mathbb{R} = E^1$

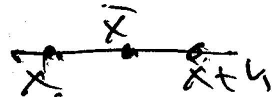
Want to show

$$|f(\vec{x}_0+\vec{h}) - f(\vec{x}_0) - \nabla f(\vec{x}_0)\cdot\vec{h}| \to 0 \text{ faster than } \vec{h}.$$

Consider $n=1$.

$$f(x_0+h) - f(x_0) - f'(x)h$$

Idea: MVT says $f(x_0+h) - f(x_0) = f'(\bar{x})h$ some $\bar{x} \in (x_0, x_0+h)$.



So $f(x_0+h) - f(x_0) - f'(x_0)h = (f'(\bar{x}) - f'(x_0))h$

where $|\bar{x} - x_0| < h$. Since f' cont, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|\bar{x} - x_0| < \delta$ then $|f'(\bar{x}) - f'(x_0)| < \varepsilon$, so

$$\frac{|f(x_0+h) - f(x_0) - f'(x_0)h|}{|h|} \le |f'(\bar{x}) - f'(x_0)| < \varepsilon \text{ if } |h| < \delta.$$

Consider $n=2$. $(\vec{x}_0 = (x_1, x_2))$ $\vec{h} = (h_1, h_2)$

$$|f(x_1+h_1, x_2+h_2) - f(x_1, x_2) - [f_x(x_1,x_2)h_1 + f_y(x_1,x_2)h_2]|$$

$$\le |f(x_1+h_1, x_2+h_2) - f(x_1, x_2+h_2) - f_x(x_1, x_2+h_2)h_1|$$

$$+ |(f_x(x_1, x_2+h_2) - f_x(x_1, x_2))h_1|$$

$$+ |f(x_1, x_2+h_2) - f(x_1, x_2) - f_y(x_1, x_2)h_2|$$

$$= ① + ② + ③$$

By MUT:

① $\le |f_x(t_1, x_2+h_2) - f_x(x_1, x_2+h_2)|\,|h_1|$

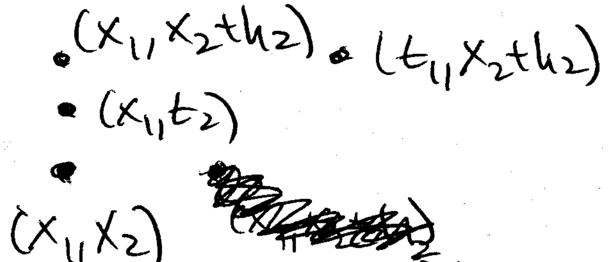
some t_1 between x_1 and x_1+h_1 so $|t_1 - x_1| < |h_1|$

② $= |f_x(x_1, x_2+h_2) - f_x(x_1, x_2)|\,|h_1|$

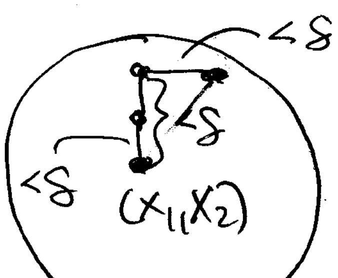
③ $\le |f_y(x_1, t_2) - f_y(x_1, x_2)|\,|h_2|$

where $|t_2 - x_2| < |h_2|$

Know: Since f_x, f_y are continuous, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if ~~$|f_x(\vec{a}) - f_x(\vec{b})|$~~ $|\vec{a} - \vec{b}| < \delta$ then $|f_x(\vec{a}) - f_x(\vec{b})| < \varepsilon$, similarly for f_y.



If I choose $\delta > 0$ small enough then if $|\vec{h}| < \delta$, all these points are in $B(\delta, (x_1, x_2))$



10.3 The Chain Rule.

A. The Product Rule (Section 10.2).

1. Theorem. Let $\mathbb{f}, \mathbb{g}: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ be differentiable at $\mathbb{x}_0 \in D$. Then

$$(\mathbb{f} \cdot \mathbb{g})'(\mathbb{x}_0) = \mathbb{f}(\mathbb{x}_0)\mathbb{g}'(\mathbb{x}_0) + \mathbb{g}(\mathbb{x}_0)\mathbb{f}'(\mathbb{x}_0)$$

2. Remark. How do we interpret this formula in terms of linear transformations?

$(f \cdot g): D \subseteq \mathbb{E}^n \to \mathbb{R} = \mathbb{E}^1$ $\quad (f \cdot g)(\vec{x}_0) = f(\vec{x}_0) \cdot g(\vec{x}_0)$

$(f \cdot g)'(\vec{x}_0) \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^1)$ so a $1 \times n$ matrix, i.e. a gradient vector.

$f'(\vec{x}_0), g'(\vec{x}_0) \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$ so a $m \times n$ matrix

$$\begin{array}{c} m \\ 1\left[\;\underset{f(\vec{x}_0)}{\longrightarrow}\;\right] \end{array} \begin{array}{c} n \\ m\left[\; g'(\vec{x}_0) \;\right] \end{array} + \begin{array}{c} m \\ 1\left[\;\longrightarrow\;\right] \end{array} \begin{array}{c} n \\ m\left[\; f'(\vec{x}_0) \;\right] \end{array} = \begin{array}{c} n \\ 1\left[\;\longrightarrow\;\right] \end{array}$$

3. Proof of Theorem:

Let $\vec{h} \in \mathbb{E}^n$. Look at

$|(f \cdot g)(\vec{x}_0 + \vec{h}) - (f \cdot g)(\vec{x}_0) - (f \cdot g)'(\vec{x}_0)\vec{h}|$. I want this to go to zero faster than $\vec{h}$. To prove Theorem, look at

$$(f \cdot g)(\vec{x}_0 + \vec{h}) - (f \cdot g)(\vec{x}_0) - (f(\vec{x}_0) g'(x_0) + g(\vec{x}_0) f'(\vec{x}_0))\vec{h}$$

$$= f(\vec{x}_0 + \vec{h}) \cdot g(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) \cdot g(\vec{x}_0) - f(\vec{x}_0) \cdot g'(\vec{x}_0)\vec{h}$$
$$- g(\vec{x}_0) \cdot f'(x_0)\vec{h}$$

$\underline{|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|}$

$$= [f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{h}] \cdot g(\vec{x}_0 + \vec{h})$$
$$+ [g(\vec{x}_0 + \vec{h}) - g(\vec{x}_0) - g'(\vec{x}_0)\vec{h}] \cdot f(\vec{x}_0)$$
$$+ [g(\vec{x}_0 + \vec{h}) - g(\vec{x}_0)] \cdot f'(\vec{x}_0)\vec{h}. = ① + ② + ③$$

$$\frac{\|①\|}{\|\vec{h}\|} \leq \frac{\|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{h}\|}{\|\vec{h}\|} \underbrace{\|g(\vec{x}_0 + \vec{h})\|}$$

is bounded for $\|\vec{h}\| \leq 1$ say since g cont at $\vec{x}_0$.

So $\dfrac{|①|}{\|\vec{h}\|} \to 0$ as $\|\vec{h}\| \to 0$.

That $\dfrac{|②|}{\|\vec{h}\|} \to 0$ as $\vec{h} \to 0$ is similar.

For ③: $$\frac{|③|}{\|\vec{h}\|} \le \|g(\vec{x}_0+\vec{h}) - g(\vec{x}_0)\|_{E^m} \frac{\|\overbrace{f'(\vec{x}_0)}^{\mathcal{L}(E^n, E^m)}\vec{h}\|_{E^m}}{\|\vec{h}\|_{E^n}}$$

$$\le \|g(\vec{x}_0+\vec{h}) - g(\vec{x}_0)\|_{E^m} \frac{\|f'(\vec{x}_0)\| \cancel{\|\vec{h}\|}_{E^n}}{\cancel{\|\vec{h}\|}_{E^n}}$$

$\to 0$ as $\vec{h} \to \vec{0}$ by continuity of g.