10.4 Hint

$T \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$. If I can show that $\forall \vec{x} \in \mathbb{E}^n$

$\boxed{\|Tx\|_{\mathbb{E}^m} \le C \|x\|_{\mathbb{E}^n}}$ for some $C > 0$

then we know $\|T\|_{\mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)} \le C$.

because $\|T\|_{\mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)}$ is the inf of all such C.

10.1 $\|L\vec{x}\| \le K \|\vec{x}\| \quad \forall x$

inf of all K

I will get $\|L\|$

Q: Does inequality still hold?

$\{K : \|L\vec{x}\| \le K \|\vec{x}\| \ \forall \vec{x} \in \mathbb{E}^n\} \subseteq \mathbb{R}$

$\inf\{-\}$ need not be in the set.

Given any $\vec{x}$, $\dfrac{\|L\vec{x}\|}{\|\vec{x}\|}$ is a lower bound of $\{-\}$.

3. Remark. (a) Differentiability of $\mathbb{f}$ at $\mathbb{x}$ implies that all of the partial derivatives of $\mathbb{f}$ exist at $\mathbb{x}$. However, the existence of all partial derivatives at $\mathbb{x}$ does not guarantee that $\mathbb{f}$ is differentiable at $\mathbb{x}$. This is in contrast to the case of real-valued functions of a single variable.

(b) However, if all of the partials of $\mathbb{f}$ are *continuous* then the story is different.

4. Theorem (10.2.3) Let $\mathbb{f}: D \to \mathbb{E}^m$, D an open subset of $\mathbb{E}^n$. Then $\mathbb{f} \in C^1(D, \mathbb{E}^m)$, that is, considering $\mathbb{f}'$ is continuous as a function $\mathbb{f}': D \to \mathcal{L}(\mathbb{E}^m, \mathbb{E}^n)$ between two normed linear spaces, if and only if every partial derivative of $\mathbb{f}$ is continuous on D.

In one variable $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists if and only if f is differentiable at x. Not true in general.

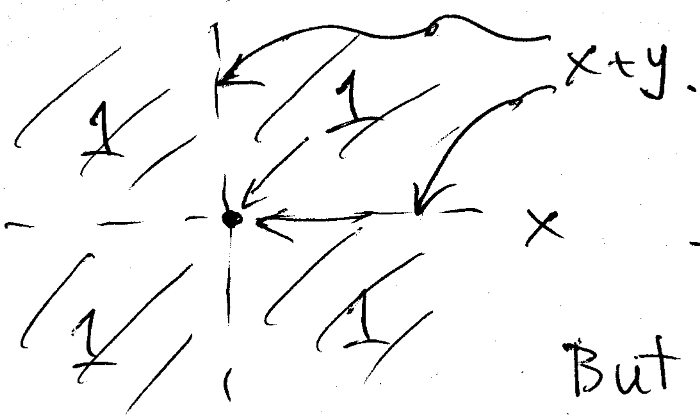
(a) If $f: \mathbb{E}^n \to \mathbb{E}^m$ is differentiable at $\vec{x}_0$ then all of its partials exist.

$$[f'(\vec{x}_0)]_{m\times n} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right]_{i=1, j=1}^{m \quad n}$$

In fact $\frac{\partial f_i}{\partial x_j}(\vec{x}_0) = \left(f'(\vec{x}_0)\vec{e}_j\right)_i$ = ith element of vector $f'(\vec{x}_0)\vec{e}_j$.

(b) If all partials of f exist at $\vec{x}_0$ then f is not necessarily differentiable there.

e.g. $f: \mathbb{E}^2 \to \mathbb{E}^1$ $\quad f(x,y) = \begin{cases} x+y & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise} \end{cases}$



Clearly.

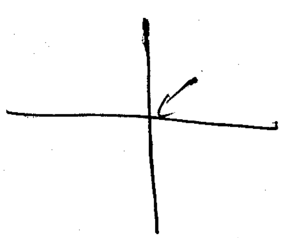
$f_x(0,0) = 1, \; f_y(0,0) = 1$

But $f'(0,0)$ does not exist. because f is not continuous at (0,0).

e.g. $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sin((x^2+y^2)^{1/2})} & 0 < (x^2+y^2)^{1/2} < \pi \\ 0 & (x,y) = (0,0) \end{cases}$

1. $f(x,y)$ is continuous at (0,0).

$$f(r,\theta) = \frac{r^2}{\sin(r)} = \overset{1}{\frac{r}{\sin(r)}} \cdot \overset{0}{r} \text{ as } r \to 0.$$



2. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{h^2}{\sin(|h|)\cdot h} = 1$

$f_y(0,0) = 1$ also.

3. Is f differentiable at $(0,0)$? No

If it were then $f'(0,0) = \nabla f(0,0) = [1, 1]$

look at

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - \cancelto{0}{f(0,0)} - [1,1]\cdot(h_1,h_2)|}{(h_1^2+h_2^2)^{1/2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \left| \underbrace{\frac{(h_1^2+h_2^2)^{1/2}}{\sin(h_1^2+h_2^2)^{1/2}}}_{\to 1} \cdot \frac{1}{\cancel{(h_1^2+h_2^2)^{1/2}}} - \underbrace{\frac{h_1+h_2}{(h_1^2+h_2^2)^{1/2}}}_{\text{does not converge.}} \right|$$

$\to 1$ along pos x-axis

$\to -1$ along neg x-axis

<u>Remark</u>: $f' \in C(D, \mathbb{E}^m)$ means what?

$$f' : D \to \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$$
$$\vec{x} \mapsto f'(\vec{x}) \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$$

Continuous ~~means~~ at $\vec{x}_0 \in D$ means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \vec{x} \in D$, $\|\vec{x} - \vec{x}_0\|_{\mathbb{E}^n} < \delta$ then $\|f'(\vec{x}_0) - f'(\vec{x})\|_{\mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)} < \varepsilon$

$\|f(\vec{x} + \vec{h}) - f(\vec{x}) - A\vec{h}\|_{\mathbb{E}^m} \to 0$ as $\vec{h} \to 0$ always

$f : \mathbb{E}^n \to \mathbb{E}^m$

$A \in \mathcal{L}(\mathbb{E}^n, \mathbb{E}^m)$

but I want $\dfrac{\|f(\vec{x} + \vec{h}) - f(\vec{x}) - A\vec{h}\|}{\|\vec{h}\|} \to 0$ as $\vec{h} \to 0$

This only works for <u>one</u> A, we call $A = f'(\vec{x})$.

Pf: (10.2.3)

($\Rightarrow$) Suppose $f' \in C(D, E^m)$. Want to show that $\frac{\partial f_i}{\partial x_j}$ is continuous on D. Note that $\frac{\partial f_i}{\partial x_j} : D \overset{\subseteq E^n}{\rightarrow} \mathbb{R}$. ~~Need to show that~~

Let $\vec{x}_0 \in D$. Want to show

$$\lim_{\vec{x} \to \vec{x}_0} \left| \frac{\partial f_i}{\partial x_j}(\vec{x}) - \frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right| = 0.$$

We know $f'(\vec{x}_0)$ exists so it can be represented by matrix of partials.

$$\underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{f'(\vec{x})} \begin{bmatrix} | \\ \vec{e}_j \\ | \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{bmatrix}$$

i^{th} comp is $\frac{\partial f_i}{\partial x_j}$

$$\begin{array}{r} \\ j^{th} \rightarrow \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Hence,

$$\left|\frac{\partial f_i}{\partial x_j}(\vec{x}) - \frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right| \le \| f'(\vec{x})\vec{e}_j - f'(\vec{x}_0)\vec{e}_j \|_{E^m}$$

$$= \|(f'(\vec{x}) - f'(\vec{x}_0))\vec{e}_j\|_{E^m} \le \| f'(\vec{x}) - f'(\vec{x}_0)\|_{\mathcal{L}(E^n, E^m)} \overbrace{\|\vec{e}_j\|_{E^n}}^{1}$$

Therefore $\left|\frac{\partial f_i}{\partial x_j}(\vec{x}) - \frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right| \to 0$ as $\vec{x} \to \vec{x}_0$

since $\|f'(\vec{x}) - f'(\vec{x}_0)\| \to 0$ as $\vec{x} \to \vec{x}_0$.

($\Leftarrow$) We have to show ① that $f'(\vec{x}_0)$ exists for $\vec{x}_0 \in D$ and ② $f' \in C(D, E^m)$. Look at ② first. We have to show $\|f'(\vec{x}) - f'(\vec{x}_0)\| \to 0$ as $\vec{x} \to \vec{x}_0$ for each $\vec{x}_0 \in D$. We have seen that if $A \in \mathcal{L}(E^n, E^m)$ represented by $(a_{ij})_{i=1,j=1}^{m\ \ n}$ then

$$\|A\|_{\mathcal{L}(E^n, E^m)} \le \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}.$$

A matrix representation of $f'(\vec{x}) - f'(\vec{x}_0)$ is given by $\left(\frac{\partial f_i}{\partial x_j}(\vec{x}) - \frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right)_{i=1,j=1}^{m\ \ n}$. Then

$$\|f'(\vec{x}) - f'(\vec{x}_0)\| \le \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}(\vec{x}) - \frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right|^2\right)^{1/2}$$

Since each $\frac{\partial f_i}{\partial x_j}$ is continuous, the right side $\to 0$ as $\vec{x} \to \vec{x}_0$. Hence $f' \in C(D, E^m)$.

($\Longleftarrow$) Want to show $f'(\vec{x}_0)$ exists for each $\vec{x}_0 \in D$.
So we will show that $[f'(\vec{x}_0)] = \left[\frac{\partial f_i}{\partial x_j}\right]$. Look at
$\left\| f(\vec{x}_0+\vec{h}) - f(\vec{x}_0) - \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right]\vec{h} \right\|_{E^m}$. I can write
this ~~as~~ difference as

$$\begin{bmatrix} f_1(\vec{x}_0+\vec{h}) - f_1(\vec{x}_0) - \nabla f_1(\vec{x}_0)\cdot\vec{h} \\ \vdots \\ f_m(\vec{x}_0+\vec{h}) - f_m(\vec{x}_0) - \nabla f_m(\vec{x}_0)\cdot\vec{h} \end{bmatrix}$$ so look at each

component separately. So assume that
we are dealing with real-valued function on E^n.
(Finish next time)