$f: D \to \mathbb{E}^m$
$D \subseteq \mathbb{E}^n$

$f \in C(D) \Leftrightarrow$
$\forall \mathcal{O} \subseteq \mathbb{E}^m$ open
$f^{-1}(\mathcal{O}) \subseteq D$ is
relatively open

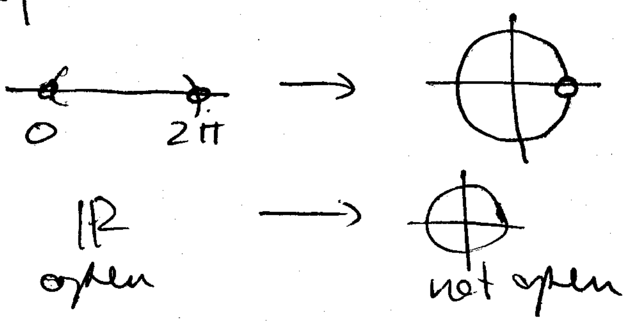
Remark. (a) Note that it is not necessarily true that the forward image of an open set by a continuous function is open.

$f: \mathbb{R} \to \mathbb{R}^2$

$t \mapsto (\cos t, \sin t)$

$f(x) = \sin x$

$\mathbb{R} \to [-1, 1]$.




(b) Nor is it necessarily true that the forward image of a closed set by a continuous function is closed.

$f(x) = \frac{1}{x}$

$[1, \infty) \to (0, 1]$

closed — not closed

$f(x) = \arctan(x)$

$\mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

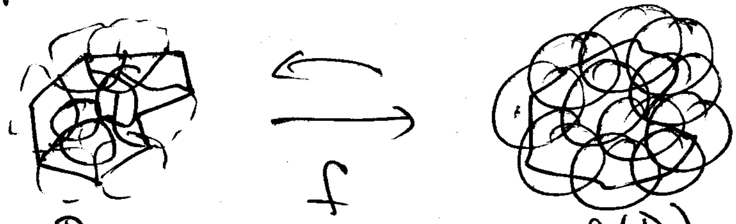
closed — open and not closed.

**Theorem** (9.3.1). Let $\mathbb{f} \in C(D)$, $\mathbb{f}: D \to \mathbb{E}^m$. If $D$ is compact, then $\mathbb{f}(D) = \{\mathbb{f}(\mathbb{x}): \mathbb{x} \in D\}$ is compact.

In fact, if $f: \mathbb{R} \to \mathbb{R}$ is continuous $f([a,b]) = [c,d]$.

$f^{-1}(S) = \{x: f(x) \in S\}$, $x \in D$

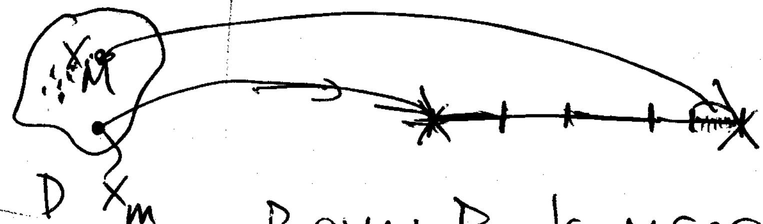
Pf: Suppose $D$ compact. Show $f(D)$ is compact. Let $\{O_\alpha\}$ be an open cover of $f(D)$. We want to extract a finite subcover. Consider the collection $\{f^{-1}(O_\alpha)\}$. Since $f \in C(D)$, each $f^{-1}(O_\alpha)$ is relatively open in $D$.




Hence there are open subsets of $\mathbb{E}^n$, call them $\{U_\alpha\}$ such that $f^{-1}(O_\alpha) = U_\alpha \cap D$. Claim: $\{U_\alpha\}$ is an open cover of $D$, i.e. $D \subseteq \bigcup_\alpha U_\alpha$. Let $x \in D$ then $f(x) \in f(D)$ so $\exists\, \alpha_0$ such that $f(x) \in O_{\alpha_0}$ so that $x \in f^{-1}(O_{\alpha_0}) \subseteq U_{\alpha_0} \subseteq \bigcup_\alpha U_\alpha$. Since $D$ is compact there exist $\alpha_1, \ldots, \alpha_N$ such that $D \subseteq \bigcup_{i=1}^N U_{\alpha_i}$.

Claim: $f(D) \subseteq \bigcup_{i=1}^N O_{\alpha_i}$. Let $y \in f(D)$. Then for some $x \in D$, $f(x) = y$. Also there is an $i_0$ such that $x \in U_{i_0}$. Hence $x \in U_{i_0} \cap D = f^{-1}(O_{i_0})$, and so $f(x) \in O_{i_0} \subseteq \bigcup_{i=1}^N O_{\alpha_i}$. So $y \in \bigcup_{i=1}^N O_{\alpha_i}$. Hence $f(D)$ is compact.

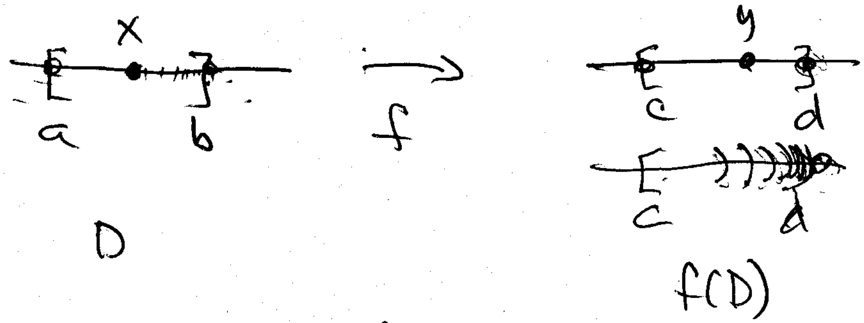
<u>Theorem</u> (Extreme Value Theorem). If $D \subseteq \mathbb{E}^n$ is compact and if $\mathbb{f} \in C(D)$, is a real-valued function, then $\mathbb{f}$ achieves its maximum and minimum values on $D$. In other words, there exist points $\mathbb{x}_M$ and $\mathbb{x}_m \in D$ such that $\mathbb{f}(\mathbb{x}) \leq \mathbb{f}(\mathbb{x}_M)$ and $\mathbb{f}(\mathbb{x}) \geq f(\mathbb{x}_m)$ for all $\mathbb{x} \in D$.




<u>Rem</u>: Book uses <u>fact that a compact set is closed & bounded</u>: $f(D)$ cpt $\Rightarrow f(D)$ bounded $\Rightarrow \exists\, M = \sup f(D)$, $m = \inf f(D)$. $f(D)$ closed $\Rightarrow m$ and $M$ are achieved.

<u>Pf</u>: (1) <u>B-W property</u>: Since $D$ is compact, so is $f(D)$ hence it is bounded. Therefore $m = \inf f(D)$ and $M = \sup f(D)$ both exist. Will show that for some $\vec{x}_M \in D$, $f(\vec{x}_M) = M$. Since $M = \sup f(D)$ there is a sequence $y_k \in f(D)$ such that $y_k \to M$. Let $\vec{x}_k \in D$ satisfy $f(\vec{x}_k) = y_k$. Since $\vec{x}_k \in D$, by B-W there is an $\vec{x}_0 \in D$ such that $x_{k_j} \to \vec{x}_0$ for some subsequence $x_{k_j}$. Claim: $f(\vec{x}_0) = M$. Since $f$ is continuous, $\vec{x}_{k_j} \to \vec{x}_0$ implies that $f(\vec{x}_{k_j}) \to f(\vec{x}_0)$. But $f(\vec{x}_{k_j}) = y_{k_j}$ and $y_{k_j} \to M$. Hence $f(\vec{x}_0) = M$.

(2): <u>H-B property</u>: Again since $f(D)$ is bounded, $m = \inf f(D)$ and $M = \sup f(D)$ exist. Will show that for some $\vec{x}_M \in D$, $f(\vec{x}_M) = M$. Suppose that

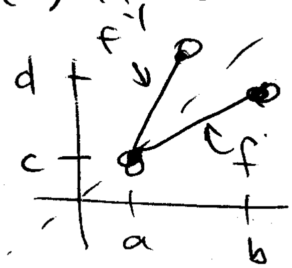


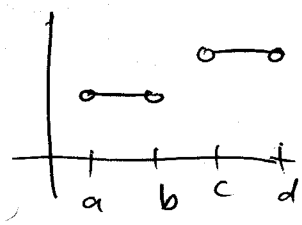

for no $\vec{x} \in D$, $f(\vec{x}) = M$. Then for all $\vec{x} \in D$, $f(\vec{x}) < M$. Consider the open sets $\mathcal{O}_k = (-\infty, M - \frac{1}{k})$. Claim 1: $f(D) \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k$. Claim 2: ~~f(D)~~ $\{\mathcal{O}_k\}$ admits no finite sub cover of $f(D)$.

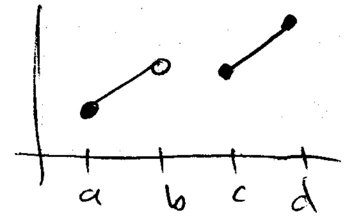
Details are an exercise.

**Theorem** (9.3.3) (Open Mapping Theorem.)
Let $\mathbb{f} \in C(D)$, $\mathbb{f}: D \to \mathbb{E}^m$, $\mathbb{f}$ is one-to-one. If $D$ is compact then $\mathbb{f}^{-1}$ is continuous.

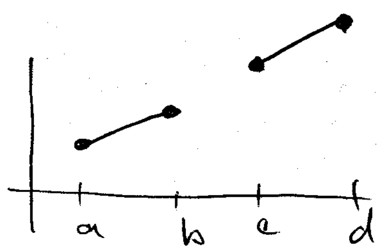
Rem: (a) $f$ is one-to-one implies $f^{-1}$ exists as a function.
(b) If $D$ is not compact, $f^{-1}$ need not be continuous.



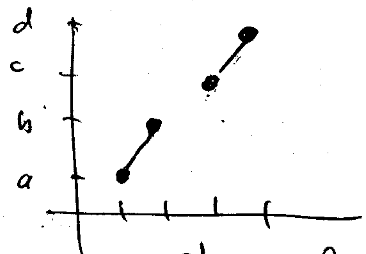



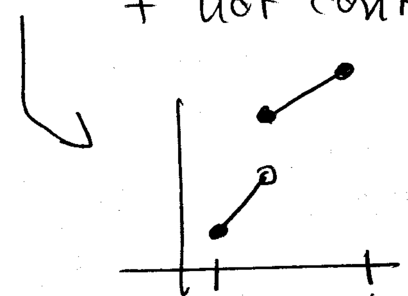

$f(x)$ 1-1 on $[a,b) \cup [c,d]$
$f^{-1}$ not cont.




$f$ cont, 1-1 on $[a,b] \cup [c,d]$



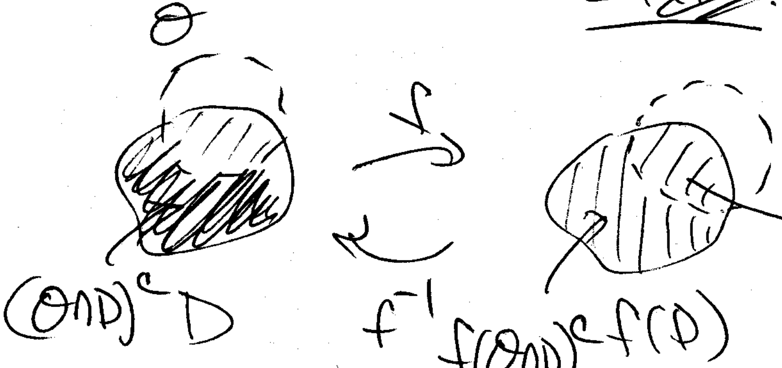

$f^{-1}$ is also continuous



not cont.

Pf: An idea: If we can show $f$(open set) is open then this is $(f^{-1})^{-1}$(open) is open, so $f^{-1}$ continuous.
$f: D \subseteq \mathbb{E}^n \to \mathbb{E}^m$ so $f: D \to f(D)$ $f^{-1}: f(D) \subseteq \mathbb{E}^m \to \mathbb{E}^n$
Let $\mathcal{O} \subseteq \mathbb{E}^n$ be open. Consider $(f^{-1})^{-1}(\mathcal{O})$. Want to show this is relatively open in $f(D)$. Claim: $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O})$ (exercise). So must show $f(\mathcal{O})$ is open in $f(D)$ if $\mathcal{O} \subseteq \mathbb{E}^n$ is open. Actually since $\mathcal{O}$ need not be subset of $D$, we are considering $f(\mathcal{O} \cap D)$.

Actually we will show that $f(\mathcal{O}\cap D)^c$ is relatively closed in $f(D)$. ~~Claim.~~ ~~$f(\mathcal{O}\cap D)^c$~~ Really looking at $f(\mathcal{O}\cap D)^c \cap f(D)$.




$f(\mathcal{O}\cap D)$ and showing it is relatively closed.

<u>Claim:</u> $f(\mathcal{O}\cap D)^c = f((\mathcal{O}\cap D)^c) = f(\mathcal{O}^c \cup D^c)$

$= f(\mathcal{O}^c) \cap f(D) = f(\mathcal{O})^c \cap f(D)$

Ugh... A correct proof follows.

Proof: The idea is to show that $f^{-1}$ is continuous by showing that the inverse image under $f^{-1}$ of an open set in $\mathbb{E}^n$ is relatively open in $f(D)$, the domain of $f^{-1}$. We will need several claims whose proofs are exercises.

Claim 1: If $f: D \to f(D)$ is one-to-one then given $A, B \subseteq D$, $f(A \cap B) = f(A) \cap f(B)$.

Claim 2: If $f: D \to f(D)$ is one-to-one then given $A \subseteq D$, $f(D - A) = f(D) - f(A)$

Claim 3: If $f: D \to f(D)$ is one-to-one then given $A \subseteq \mathbb{E}^n$, $(f^{-1})^{-1}(A) = f(A \cap D)$.

---

Let $\mathcal{O} \subseteq \mathbb{E}^n$ be open. Then $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O} \cap D)$. We must show that $f(\mathcal{O} \cap D)$ is relatively open in $f(D)$. By a homework exercise it is enough to show that $f(D) - f(\mathcal{O} \cap D)$ is relatively closed in $f(D)$. Since $\mathcal{O} \cap D$ is relatively open in $D$, $D - (\mathcal{O} \cap D) = \mathcal{O}^c \cap D$ is relatively closed in $D$ and since $D$ is compact, $f(\mathcal{O}^c \cap D)$ is compact (since $\mathcal{O}^c \cap D$ is a closed subset of a compact set) and hence closed in $\mathbb{E}^m$. Since $f(\mathcal{O}^c \cap D) \subseteq f(D)$ it is a closed subset of $f(D)$. But applying our claims,
$f(D) - f(\mathcal{O} \cap D) = f(D - (\mathcal{O} \cap D)) = f(D - \mathcal{O}) = f(\mathcal{O}^c \cap D)$
Hence $f(D) - f(\mathcal{O} \cap D)$ is relatively closed in $f(D)$.