Theorem 1. (9.1.1) Suppose that $\mathbb{a}$ is a limit point of the domain $D_{\mathbb{f}}$ of the function $\mathbb{f}$. Then the following are equivalent

a. $\lim_{\mathbb{x}\to\mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{L}$.

b. For every sequence $\{\mathbb{x}^{(j)}\} \in D_{\mathbb{f}}$, with $\mathbb{x}^{(j)} \neq \mathbb{a}$ for all j, such that $\mathbb{x}^{(j)} \to \mathbb{a}$, $\lim_{j\to\infty} \mathbb{f}\left(\mathbb{x}^{(j)}\right) = \mathbb{L}$.

Proof. ($\Longrightarrow$) Last time

($\Longleftarrow$) Assume it is not true that $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$.

We must find a sequence $\vec{x}^{(j)} \in D_f$, $\vec{x}^{(j)} \neq \vec{a}$ all j, $\vec{x}^{(j)} \to \vec{a}$ but $f(\vec{x}^{(j)}) \not\to L$. Since (a) fails there is an $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is an $\vec{x} \in D_f$ such that $0 < \|\vec{x} - \vec{a}\| < \delta$ but $\|f(\vec{x}) - L\| \geq \varepsilon_0$. Let $\vec{x}^{(j)} \in D_f$ correspond to $\delta = \frac{1}{j}$. Then $0 < \|\vec{x}^{(j)} - \vec{a}\| < \frac{1}{j}$, and $\|f(\vec{x}^{(j)}) - L\| \geq \varepsilon_0$. But this implies that $\vec{x}^{(j)} \neq \vec{a}$ for all j and $\vec{x}^{(j)} \to \vec{a}$, and that $f(\vec{x}^{(j)}) \not\to L$.

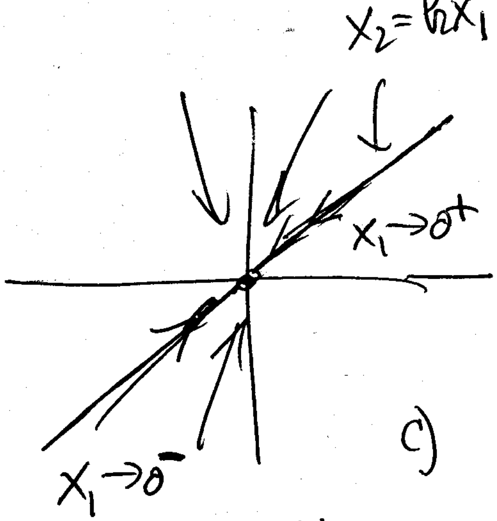
e.g. 9.4) $f(x_1,x_2) = \begin{cases} \dfrac{x_1^2 x_2}{x_1^4 + x_2^2} & (x_1,x_2) \neq (0,0) \\ 0 & x_1 = x_2 = 0. \end{cases}$

$\lim_{\vec{x} \to (0,0)} f(x_1,x_2) = L \quad \forall \varepsilon, \exists \delta$ s.t. $|x_1^2 + x_2^2|^{1/2} < \delta$ ($\|\vec{x} - \vec{0}\|$)

then $\left| \dfrac{x_1^2 x_2}{x_1^4 + x_2^2} \right| < \varepsilon.$

If $x_2 = k x_1$: $f(x_1,x_2) = f(x_1, k x_1) = \dfrac{k x_1^3}{x_1^4 + k^2 x_1^2}$

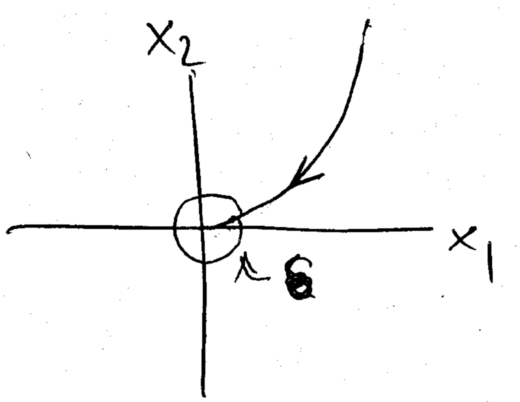
$$= \frac{k x_1}{x_1^2 + k^2} \longrightarrow 0 \text{ as } x_1 \to 0.$$



$\lim_{(x_1,x_2) \to (0,0)} f(x_1,x_2) = 0$ becomes a 1-sided limit.

c) $x_2 = x_1^2$ $\quad f(x_1,x_2) = f(x_1, x_1^2)$

$$= \frac{x_1^4}{x_1^4 + x_1^4} = \frac{1}{2}$$



Show $\lim\limits_{(x_1,x_2)\to(0,0)} f(x_1,x_2)$ dne use sequential char:

eg let $x^{(j)} = (\frac{1}{j}, 0)$ so $\vec{x}^{(j)} \to (0,0)$

and $f(\vec{x}^{(j)}) \to 0$

let $\vec{y}^{(j)} = (\frac{1}{j}, \frac{1}{j^2})$ $\vec{y}^{(j)} \to (0,0)$

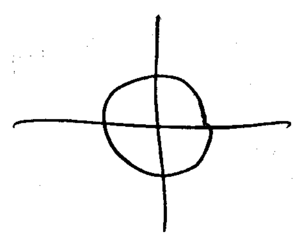
and $f(\vec{y}^{(j)}) \to \frac{1}{2}$.

(d) take $\vec{x}^{(j)} = (j, 0)$ $f(x^{(j)}) \to 0$.

$\vec{y}^{(j)} = (j, j^2)$ $f(x^{(j)}) \to \frac{1}{2}$

9.5 $$f(x_1,x_2) = \begin{cases} \dfrac{x_1 x_2^2}{x_1^4 + x_2^2} & (x_1,x_2) \neq (0,0) \\ 0 & x_1 = x_2 = 0 \end{cases}$$

$$\left|\frac{x_1 x_2^2}{x_1^4+x_2^2}\right| < \underbrace{\left|\frac{x_1}{x_1^4+x_2^2}\right|}\ |x_2|^2$$



$|x_1| \le x_1^4 + x_2^2$

$\delta = \varepsilon$

$(x_1^2+x_2^2)^{1/2} < \delta = \varepsilon \Longrightarrow \left|\frac{x_1 x_2^2}{x_1^4+x_2^2}\right| < \varepsilon$

$$\left|\frac{x_1 x_2^2}{x_1^4+x_2^2}\right| = |x_1| \left|\frac{x_2^2}{x_1^4+x_2^2}\right| \le |x_1|$$

Take $\delta = \varepsilon$. If $\underline{(x_1^2+x_2^2)^{1/2} < \varepsilon = \delta}$

then since $\underline{|x_1|} \le (x_1^2)^{1/2} \le (x_1^2+x_2^2)^{1/2} < \varepsilon$

Hence $\left|\frac{x_1 x_2^2}{x_1^4+x_2^2}\right| \le |x_1| < \varepsilon$.

4. Example. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x)\sin(y)}{x^2+y^2}$ or prove it does not exist.

5. Example. Find $\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2+2y^4}$ or prove it does not exist.

6. Example. Find $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ or prove it does not exist.

9.2/9.3 Continuous Functions.

A. Definition of Continuity.

<u>Definition.</u> Let $\mathbb{f}: D \to \mathbb{E}^m$, where $D = D_{\mathbb{f}} \subseteq \mathbb{E}^n$. Let $\mathbb{a} \in D$. We say that $\mathbb{f}$ is *continuous at* $\mathbb{a}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $\mathbb{x} \in D \cap B(\mathbb{a}, \delta)$, $\|\mathbb{f}(\mathbb{x}) - \mathbb{f}(\mathbb{a})\| < \epsilon$.

$x \in D$ s.t. $\|\vec{x} - \vec{a}\| < \delta$

$f(\vec{x}) \in B(f(\vec{a}), \varepsilon)$

We say that $\mathbb{f}$ is *continuous on* D if it is continuous at every point of D and we write $\mathbb{f} \in C(D)$.

Put this on final

<u>Theorem.</u> A function $\mathbb{f}$ is continuous at a limit point $\mathbb{a} \in D_{\mathbb{f}}$ if and only if

$$\lim_{\mathbb{x}\to\mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{f}(\mathbb{a}).$$

Proof. $(\Longrightarrow)$ Suppose f is continuous at $\vec{a}$. Let $\varepsilon > 0$, and choose δ in the definition of continuity. That is, if $\vec{x} \in D$ and $\|\vec{x} - \vec{a}\| < \delta$ then $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$. In particular, if $\vec{x} \in D$ and $0 < \|\vec{x} - \vec{a}\| < \delta$, $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$. Hence $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

$(\Longleftarrow)$ Suppose $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$. and let $\varepsilon > 0$. Choose $\delta > 0$ such that if $\vec{x} \in D$ and $0 < \|\vec{x} - \vec{a}\| < \delta$ then $\|f(\vec{x}) - f(\vec{a})\| < \varepsilon$. But if $\vec{x} = \vec{a}$ then $\|f(\vec{x}) - f(\vec{a})\| = 0 < \varepsilon$. Hence f is continuous at $\vec{a}$.

<u>Definition.</u> A set $E \subseteq \mathbb{E}^n$ is *relatively open in* $D \subseteq \mathbb{E}^n$ if there is an open set U in $\mathbb{E}^n$ such that $E = U \cap D$. E is *relatively closed in* D if there is a closed set $F \subseteq \mathbb{E}^n$ such that $E = F \cap D$.

<u>Remark</u>. (a) Note that for E to be relatively open or relatively closed in D it must be true that $E \subseteq D$.

(b) The concept of *open* and *closed* in a normed vector space is dependent on the space in which the set is assumed to sit. In other words, a set E is not simply open or closed, but is only open or closed *as a subset of some other set.*

(c) For example, the set $[0,1]$ is not open as a subset of $\mathbb{R}$ (the ambient vector space), but is open as a subset of $[0,1]$, that is if we assume that our ambient vector space is $[0,1]$.

~~Let $x \in [0,1]$~~ Let $U = \mathbb{R}$ then $E = [0,1] = \mathbb{R} \cap [0,1]$ and $\mathbb{R}$ is open.

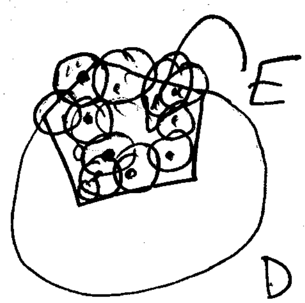
If $x \in [0,1]$ then if $x \in (0,1)$, $\exists\, \varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq (0,1)$. If $x = 0$ then $\{y \in [0,1] : |x-y| < \varepsilon\} \subseteq [0,1]$.

Theorem (9.2.3). A subset $E \subseteq D$ is relatively open in D if and only if for every $\mathbb{x} \in E$, there is a $\delta > 0$ such that $B(\mathbb{x}, \delta) \cap D \subseteq E$.

Proof. ($\Longrightarrow$) $E \subseteq D$ relatively open. Then $\exists$ U open such that $E = U \cap D$. Given $x \in E$, $x \in U$, so $\exists \delta$ st. $B(x, \delta) \subseteq U$ so $B(x, \delta) \cap D \subseteq U \cap D = E$

($\Longleftarrow$) Must find U open such that $E = U \cap D$.

$$U = \bigcup_{x \in E} B(x, \delta)$$



$U, \mathbb{E}^n$

Theorem. (9.2.4). Let $\mathbb{f}: D \to \mathbb{E}^m$. Then $\mathbb{f} \in C(D)$ if and only if for every open set U in $\mathbb{E}^m$, the inverse image

$$\mathbb{f}^{-1}(U) = \{\mathbb{x} \in D : \mathbb{f}(\mathbb{x}) \in U\}$$

is relatively open in D.

($\Rightarrow$) Suppose $f \in C(D)$ and let $U \subseteq \mathbb{E}^m$ be open. Let $\vec{x}_0 \in f^{-1}(U)$. We must find $\delta > 0$ such that $B(\vec{x}_0, \delta) \cap D \subseteq f^{-1}(U)$. Since $\vec{x}_0 \in f^{-1}(U)$, $f(\vec{x}_0) \in U$ so for some $\varepsilon > 0$, $B(f(\vec{x}_0), \varepsilon) \subseteq U$. For this $\varepsilon > 0$ choose δ so that if $\vec{x} \in B(\vec{x}_0, \delta) \cap D$ then $\|f(\vec{x}_0) - f(\vec{x})\| < \varepsilon$, that is, $B(f(\vec{x}_0), \varepsilon) \ni f(\vec{x})$. But this means $f(\vec{x}) \in U$ i.e. $\vec{x} \in f^{-1}(U)$, so $f^{-1}(U)$ relatively open.

($\Leftarrow$) Let $\vec{x}_0 \in D$, $\varepsilon > 0$. We need to find $\delta > 0$ s.t. if $\vec{x} \in D$ and $\|\vec{x} - \vec{x}_0\| < \delta$ then $\|f(\vec{x}) - f(\vec{x}_0)\| < \varepsilon$. Consider $B(f(\vec{x}_0), \varepsilon)$. This set is open in $\mathbb{E}^m$ so $f^{-1}(B(f(\vec{x}_0), \varepsilon))$ is relatively open in D. Also $\vec{x}_0 \in f^{-1}(B(f(\vec{x}_0), \varepsilon))$. So there is a $\delta > 0$ such that $B(\vec{x}_0, \delta) \subseteq f^{-1}(B(f(\vec{x}_0), \varepsilon))$. This means if $\vec{x} \in B(\vec{x}_0, \delta)$ and $\vec{x} \in D$ then $f(\vec{x}) \in B(f(\vec{x}_0), \varepsilon)$, that is, $\|f(\vec{x}) - f(\vec{x}_0)\| < \varepsilon$.