Remark: If we want to show

closed + bounded $\Longrightarrow$ compact, here is what we do!

E is bounded $\Longrightarrow$ $E \subseteq \underbrace{\overline{B(\vec{0},R)}}_{\text{this is compact.}}$ some $R>0$.

E is closed $\Longrightarrow$ E is closed subset of compact set. $\Longrightarrow$ E compact

HW problem.

9. <u>Theorem</u>. Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent.

a. A is compact.
b. A is closed and bounded.
c. Every sequence of points in A has a limit point $\boxed{\text{in } A}$, that is, every sequence in A has a convergent subsequence. (B-W)

Proof: We already showed (a) $\Leftrightarrow$ (b)
We must show (b) $\Leftrightarrow$ (c).
(b) $\Rightarrow$ (c) Assume A is closed and bounded and let $\vec{x}^k$ be a <u>sequence</u> in A. Since A is bounded, $A \subseteq B(\vec{0}, R)$ for some $R>0$. This means $\vec{x}^k$ is a bounded sequence. By B-W $\vec{x}^k$ has a convergent subsequence, say $\vec{x}^{k_j} \to \vec{x}$ as $j \to \infty$. It remains to show that $\vec{x} \in A$. Suppose $\vec{x} \notin A$. Then since A is closed, $\vec{x}$ is not a limit point of A. But given $\varepsilon > 0$ there is a J such that $j \geq J$ implies $\vec{x}^{k_j} \in B(\vec{x}, \varepsilon)$. But since $\vec{x}^{k_j} \in A$ this implies $\vec{x}$ is a limit point of A. Hence $\vec{x} \in A$.

E^n is closed and not bounded.
$[0,\infty)$ is closed in $\mathbb{R}$ but not open

(c) $\Rightarrow$ (b) Suppose (b) does not hold. Then A is either not closed or not bounded. If A is not bounded then for every $n \in \mathbb{N}$ there is an $\vec{x}^n \in A$ such that $\|\vec{x}^n\| \geq n$. Hence every subsequence of $\vec{x}^n$ is unbounded hence not convergent. So (c) fails to hold. If A is not closed then there is an $\vec{x}$ such that $\vec{x} \notin A$ and $\vec{x}$ is a limit point of A. Then for each $n \in \mathbb{N}$ there is an $\vec{x}^n \in A$ such that $0 < \|\vec{x}^n - \vec{x}\| < \frac{1}{n}$. This means $\vec{x}^n \to \vec{x}$ and hence every subsequence of $\vec{x}^n \to \vec{x}$ but $\vec{x} \notin A$. Thus (c) fails again.

9.1 Limits of Functions.

A. Definition of limit.

$f: \mathbb{E}^n \to \mathbb{E}^m$, $\mathbb{E}^n \supseteq D$

1. Definition. We will consider *vector-valued functions*, $\mathbb{f}: D \to \mathbb{E}^m$, with domain $D = D_{\mathbb{f}} \subseteq \mathbb{E}^n$. We write
$$\mathbb{f}(\mathbb{x}) = \mathbb{f}(x_1, \dots, x_n) = (f_1(\mathbb{x}), f_2(\mathbb{x}), \dots, f_m(\mathbb{x}))$$
where $f_i: D \to \mathbb{R}$ and we usually write
$$f_i(\mathbb{x}) = f_i(x_1, x_2, \dots, x_n).$$

2. Definition. Let a be a limit point (cluster point) of the domain $D_{\mathbb{f}}$ of a function $\mathbb{f}$. Then
$$\lim_{\mathbb{x} \to \mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{L}$$
if for all $\epsilon > 0$ there is an $\delta > 0$ such that for all $\mathbb{x} \in D_{\mathbb{f}}$, if $0 < \|\mathbb{x} - \mathbb{a}\| < \delta$, then $\|\mathbb{f}(\mathbb{x}) - \mathbb{L}\| < \epsilon$.

Note that $\vec{a}$ need not be in D, so $f(\vec{a})$ need not be defined.

3. Remark. If $\mathbb{a}$ is an isolated point of $D_{\mathbb{f}}$ then it does not make sense to talk about
$$\lim_{\mathbb{x} \to \mathbb{a}} \mathbb{f}(\mathbb{x}).$$

why? If $\vec{a}$ is an isolated point of D_f then given $\varepsilon > 0$, choose $\delta > 0$ such that $0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \vec{x} \notin D_f$. Hence $\forall \vec{x} \in D_f$, $0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - L\| < \varepsilon$ holds no matter what L is.

Theorem 1. (9.1.1) Suppose that $\mathbb{a}$ is a limit point of the domain $D_{\mathbb{f}}$ of the function $\mathbb{f}$. Then the following are equivalent

a. $\lim_{\mathbb{x}\to\mathbb{a}} \mathbb{f}(\mathbb{x}) = \mathbb{L}$.

b. For every sequence $\{\mathbb{x}^{(j)}\} \in D_{\mathbb{f}}$, with $\mathbb{x}^{(j)} \neq \mathbb{a}$ for all j, such that $\mathbb{x}^{(j)} \to \mathbb{a}$, $\lim_{j\to\infty} \mathbb{f}\left(\mathbb{x}^{(j)}\right) = \mathbb{L}$.

Proof. (a) $\Rightarrow$ (b) Suppose $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ and let $\vec{x}^j \in D_f$ satisfy $\vec{x}^j \to \vec{a}$ as $j \to \infty$ and $\vec{x}^j \neq \vec{a}$ for all j. We must show that $f(\vec{x}^j) \to L$ as $j \to \infty$. Since $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$, given $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < \|\vec{x}-\vec{a}\| < \delta$ then $\|f(\vec{x}) - L\| < \varepsilon$. Since $\vec{x}^j \to \vec{a}$ there is an N such that if $j \geq N$ then $\|\vec{x}^j - \vec{a}\| < \delta$ and since $\vec{x}^j \neq \vec{a}$, $0 < \|\vec{x}^j - \vec{a}\| < \delta$ for all $j \geq N$. But this implies $\|f(\vec{x}^j) - L\| < \varepsilon$ for all $j \geq N$. Hence $f(\vec{x}^j) \to L$ as $j \to \infty$.

(b) $\Rightarrow$ (a) Next time.