5.46 $g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$

Know: $\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ $x \in (-1,1)$ (ie. $|x|<1$).

Q: what happens at $x = \pm 1$?

① $x=1$: $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = g(1)$ converges.

② $x=-1$ $g(-1) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ converges.

$\therefore \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ converges for each $x \in [-1,1]$.

Does it converge uniformly on $[-1,1]$?

Weierstass M-test:

$M_k = \sup_{-1 \le x \le 1} \left| \frac{(-1)^k}{2k+1} x^{2k+1} \right| = \frac{1}{2k+1}$ no good.

Go back to looking at:

$\sup_{x \in [-1,1]} \left| \sum_{k=m}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} \right|$

use: $\sum_{k=1}^{\infty} (-1)^k a_k$ $a_n \searrow 0$

$|S_n - S| < a_{n+1}$

3. Remarks.

   a. We want to characterize the sets that have a property like that described in the B-W Theorem, specifically: *For which sets A is it true that any sequence $\{\mathbb{x}^k\} \subseteq A$ has a convergent subsequence?*

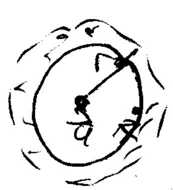
   b. B-W Theorem says that if A is a closed ball, this property holds.

   c. It is also possible to think of this property as saying: *Which subsets of $\mathbb{R}$ are most like finite sets in a particular sense?*

4. <u>Definition.</u> (8.3.1) An *open cover* of a subset $S \subseteq \mathbb{R}$ is a collection of open sets (possibly infinite) $\mathcal{O} = \{O_\alpha : \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} O_\alpha$. The set S is said to be *compact* if every open cover of S has a finite subcover, that is if $\{O_\alpha : \alpha \in A\}$ is an open cover of S then there exist indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$S \subseteq \bigcup_{k=1}^{n} O_{\alpha_k}.$$

convention
$B(a,r) = \emptyset$ if $r<0$



$B(\vec{a},r)$

5. <u>Examples.</u> (a) An open ball $B(\mathbb{a}, r)$ is not compact. $\mathcal{O}_n = B(\vec{a}, r-\frac{1}{n})$, $\{\mathcal{O}_n\}$ covers $B(\vec{a},r)$ but any finite subcover excludes some $\vec{x} \in B(\vec{a},r)$

   (b) A finite subset of $\mathbb{R}^n$ is compact.

exercise

6. Theorem. A closed ball in $\mathbb{R}^n$ is compact.

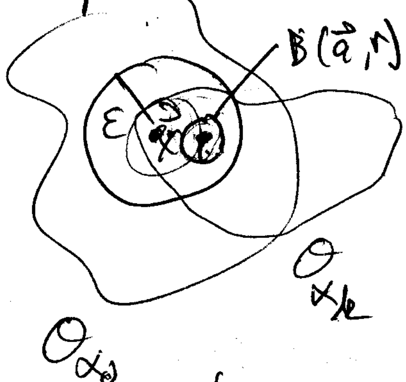
7. Claim: Any open cover of a set $A \subseteq \mathbb{R}^n$ admits a *countable* subcover.

Proof of Claim: First note that $\mathbb{Q}^n = \{(r_1, r_2, \dots, r_n) : r_i \in \mathbb{Q}\}$ is dense in $\mathbb{R}^n$. (Dense? For every $\vec{x} \in \mathbb{R}^n$, there exists a sequence $\vec{r}^k \in \mathbb{Q}^n$ such that $\|\vec{x} - \vec{r}^k\| \to 0$ as $k \to \infty$.) For $\mathbb{R}^n$, use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and given $\vec{x} = (x_1, x_2, \dots, x_n)$ find a sequence $r_j^k \in \mathbb{Q}$ such that $r_j^k \to x_j$ for each j. Then $\vec{r}^k = (r_1^k, r_2^k, \dots, r_n^k) \longrightarrow \vec{x}$ in $\mathbb{R}^n$.

Let $\{\mathcal{O}_\alpha\}$ be any collection of open sets. Want to find a sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ such that $\bigcup_{i=1}^{\infty} \mathcal{O}_{\alpha_i} = \bigcup_{\alpha} \mathcal{O}_\alpha$. Consider the collection of balls in $\mathbb{R}^n$ given by $B(\vec{a}, r)$, $\vec{a} \in \mathbb{Q}^n$, $r \in \mathbb{Q}$. There are countably many such balls. For each $B(\vec{a}, r)$, choose an α such that $B(\vec{a}, r) \subseteq \mathcal{O}_\alpha$ when possible, otherwise don't. Then index all of these α's by $\alpha_1, \alpha_2, \dots$

Claim: $\bigcup_{i=1}^{\infty} O_{\alpha_i} = \bigcup_{\alpha} O_{\alpha}$

The inclusion $\subseteq$ is clear. So let $\vec{x} \in \bigcup_{\alpha} O_{\alpha}$ then $\vec{x} \in O_{\alpha_0}$ for some α_0. Since O_{α_0} is open there is an $\varepsilon > 0$ such that $B(\vec{x}, \varepsilon) \subseteq O_{\alpha_0}$



Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, I can find $\vec{a} \in \mathbb{Q}^n$ ~~with $\|\vec{x} - \vec{a}\| < \frac{\varepsilon}{2}$.~~ ~~and an $r \in \mathbb{Q}$ with $0 < r < \frac{\varepsilon}{2}$.~~ ~~Hence $B(\vec{a}, r) \subseteq B(\vec{x}, \varepsilon) \subseteq O_{\alpha_0}$~~ and an $r \in \mathbb{Q}$ such that $\vec{x} \in B(\vec{a}, r)$ ~~and B(~~ and $B(\vec{a}, r) \subseteq O_{\alpha_0}$. Then $\vec{x}$ is in the O_{α_k} that corresponds to the $B(\vec{a}, r)$.

Pick $\vec{a}$ so that $\|\vec{x} - \vec{a}\| < \frac{\varepsilon}{4}$, $r \in \mathbb{Q}$, $\frac{\varepsilon}{4} < r < \frac{\varepsilon}{2}$. So $\vec{x} \in B(\vec{a}, r)$ and $B(\vec{a}, r) \subseteq B(\vec{x}, \varepsilon)$.

Proof of Theorem:

Let $\vec{a} \in \mathbb{R}^n$, $r>0$. Show $\overline{B(\vec{a},r)}$ is compact. Let $\{O_\alpha\}$ be an open cover of $\overline{B(\vec{a},r)}$. By Claim, we can assume that our cover is countable, i.e. $\{O_n : n \in \mathbb{N}\}$. Suppose there is no finite subcover. This means that for each $k \in \mathbb{N}$ there is an $\vec{x}_k \in \overline{B(\vec{a},r)}$ with $\vec{x}_k \notin \bigcup_{n=1}^{k} O_n$. Since $\vec{x}_k \in \overline{B(\vec{a},r)}$ it is a bounded sequence hence by B-W has a convergent subsequence say $\vec{x}_{k_j} \longrightarrow \vec{x}$. Since $\overline{B(\vec{a},r)}$ is closed, $\vec{x} \in \overline{B(\vec{a},r)}$ and so must be in a set O_N some $N \in \mathbb{N}$. Since O_N is open for some $\varepsilon > 0$, $B(\vec{x},\varepsilon) \subseteq O_N$. Since $\vec{x}_{k_j} \to \vec{x}$ there is a $K>0$ such that if $j \geq K$ then $\vec{x}_{k_j} \in B(\vec{x},\varepsilon)$. But this is a contradiction since $\vec{x}_{k_j} \notin \bigcup_{n=1}^{k_j} O_n$ hence $\vec{x}_{k_j} \notin O_N$ if j is large enough, i.e. if $k_j \geq N$. Therefore $\{O_n\}$ admits a finite subcover of $\overline{B(\vec{a},r)}$.

8. <u>Theorem.</u> (8.3.1) A set $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

<u>Proof.</u>

($\Longrightarrow$) Suppose A is compact. We will show it is closed and bounded. First we will show A^c is open. Let $\vec{x} \in A^c$. Will find $r > 0$ such that $B(\vec{x}, r) \subseteq A^c$. For each $\vec{y} \in A$, $\vec{y} \neq \vec{x}$ so $\|\vec{x} - \vec{y}\| = r_{\vec{y}} > 0$. Let $O_{\vec{y}} = B(\vec{y}, r_{\vec{y}}/2)$. Then $\{O_{\vec{y}}\}_{\vec{y} \in A}$ is an open cover of A, hence there is a finite set $\{\vec{y}_1, \dots, \vec{y}_n\}$ such that $A \subseteq \bigcup_{i=1}^{n} O_{\vec{y}_i}$

Let $r = \min\{r_{\vec{y}_i} : i = 1, \dots, n\}$. Want to show that $B(\vec{x}, \frac{r}{2}) \subseteq A^c$. Let $\vec{z} \in B(\vec{x}, \frac{r}{2})$. If $\vec{z} \in A$ then $\vec{z} \in O_{\vec{y}_i}$ for some i. Hence $\|\vec{y}_i - \vec{z}\| < \frac{r_{y_i}}{2}$

But also, $\|\vec{z} - \vec{x}\| < \frac{r}{2} \leq \frac{r_{y_i}}{2}$. Hence

$$r_{y_i} = \|\vec{x} - \vec{y}_i\| \leq \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}_i\| < \frac{r_{y_i}}{2} + \frac{r_{y_i}}{2} = r_{y_i}$$

a contradiction. Hence $\vec{z} \notin A$ so $B(\vec{x}, \frac{r}{2}) \subseteq A^c$ and A is closed.

Now lets show A is bounded. ~~Assume A is not bounded and~~ Let $\mathcal{O}_n = B(\vec{0}, n)$. Then $A \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n$ so $\{\mathcal{O}_n\}$ is an open cover of A.

Since A is compact, there is an N such that $A \subseteq \bigcup_{n=1}^{N} \mathcal{O}_n = \mathcal{O}_N = B(\vec{0}, N)$. Therefore if $\vec{x} \in A$ then $\|\vec{x}\| \leq N$. Hence A is bounded.