8.14

Idea: V vector space

inner product : $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$
satisfies Def 8.1.1.

Then define norm : $\| \cdot \| : V \to \mathbb{R}^+$

$$\| x \| = (\langle x, x \rangle)^{1/2}$$

satisfies certain axioms (Ch. 2).

There are in principle many norms on V

$\mathbb{R}^n$:

$$\|\vec{x}\|_2 = (\langle \vec{x}, \vec{x} \rangle)^{1/2} = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}$$

$$\|\vec{x}\|_1 = \sum_{j=1}^{n} |x_j| \quad \text{(taxicab norm in } \mathbb{R}^2\text{)}$$

$$\|\vec{x}\|_\infty = \max_j |x_j|$$

Any $1 < p < \infty$: $\|\vec{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$

Q: Given a norm on V, does it come from an inner product?

A: It does only if it satisfies Parallelogram Law. (In fact: if and only if)

Thm: A set is closed if and only if its complement is open.

Proof.

($\Rightarrow$) Let $S \subseteq \mathbb{R}^n$ be closed. We must show S^c is open. Let $\vec{x} \in S^c$. Will find $\varepsilon > 0$ such that $B(\vec{x}, \varepsilon) \subseteq S^c$. Since $\vec{x} \notin S$, $\vec{x}$ cannot be a limit pt. of S. This means that there is an $\varepsilon > 0$ such that for all $\vec{y} \in S$, either $\vec{y} = \vec{x}$ or $\|\vec{x} - \vec{y}\| \geq \varepsilon$. Since $\vec{x} \notin S$, $\vec{y} \neq \vec{x}$. Hence for all $\vec{y} \in S$, $\|\vec{x} - \vec{y}\| \geq \varepsilon$. But this means $B(\vec{x}, \varepsilon) \subseteq S^c$. Hence S^c is open.

$\vec{x}$ is limit pt of S means $\forall \varepsilon > 0$, $B(\vec{x}, \varepsilon) \cap S$ contains points other than $\vec{x}$, i.e. $\forall \varepsilon > 0$, $\exists \vec{y} \in S$ s.t. $0 < \|\vec{x} - \vec{y}\| < \varepsilon$.

($\Leftarrow$) Assume S^c is open. Show S contains all of its limit points. Let $\vec{x}$ be a limit point of S and assume that $\vec{x} \in S^c$. Since S^c is open there is an $\varepsilon > 0$ such that $B(\vec{x}, \varepsilon) \subseteq S^c$. This means that there is no $\vec{y} \in S$ satisfying $\|\vec{x} - \vec{y}\| < \varepsilon$. Since also $\vec{x} \neq \vec{y}$, we have $0 < \|\vec{x} - \vec{y}\| < \varepsilon$. for no $\vec{y} \in S$. Hence $\vec{x}$ is not a limit point of S.

6. Theorem. The union of finitely many closed sets is closed and the intersection of any

number of closed sets is closed. The union of infinitely many closed sets need not be closed.

<u>Idea:</u> De Morgan's Laws.

Let F_n be closed for $n=1,2,\dots,N$.

$$F = \bigcup_{n=1}^{N} F_n . \quad F^c = \left(\bigcup_{n=1}^{N} F_n\right)^c = \bigcap_{n=1}^{N} F_n^c$$

each F_n^c is open, so $\bigcap_{n=1}^{N} F_n^c$ is open, so F^c is open.

Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets

$$\text{Let } F = \bigcap_{\alpha\in A} F_\alpha . \quad F^c = \left(\bigcap_{\alpha\in A} F_\alpha\right)^c = \bigcup_{\alpha\in A} F_\alpha^c$$

which is a union of open sets, hence open.

Therefore F is closed.

Compact sets.

3 seperate but <u>related</u> concepts:

(1) B-W Thm:

(2) Heine-Borel property.

(3) Closed and Bounded.

<u>e.g.</u>: Closed and not bounded: $\mathbb{Z} \subseteq \mathbb{R}$

$\mathbb{Z}$ has no limit points since all of its points are isolated. Also $\mathbb{R}$ is closed but not bounded.

<u>e.g.</u> Bounded and not closed: $(0,1) \subseteq \mathbb{R}$ is not closed because 0 is a limit point and $0 \notin (0,1)$.

These notions are <u>equivalent</u>.

8.3. Compact Sets.

A. The Bolzano-Weierstrass Theorem.

1. Theorem. (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

2. Theorem. (Bolzano-Weierstrass) Every bounded sequence $\mathbf{x}^k$ in $\mathbb{R}^n$ has a convergent subsequence.

Proof: Idea: If $\vec{x}^k$ is bounded, this means there is an $M > 0$ such that $\|\vec{x}^k\| \leq M$ for all k. If we write $\vec{x}^k = (x_1^k, x_2^k, \dots, x_n^k)$ then we know that for each j, $|x_j^k| \leq \|\vec{x}^k\| \leq M$ all k. Hence x_j^k is a bounded sequence in $\mathbb{R}$ and so has a convergent subsequence. What to do?

$$x_1^1 \quad x_1^2 \quad x_1^3 \quad \dots$$

$$x_1^{k_1} \quad x_1^{k_2} \quad x_1^{k_3} \quad \dots$$

$$x_2^{k_1} \quad x_2^{k_2} \quad x_2^{k_3} \quad \dots$$

$$x_2^{k_{n_1}} \quad x_2^{k_{n_2}} \quad x_2^{k_{n_3}} \quad \dots$$

$$\vec{x}^1 \quad \vec{x}^2 \quad \vec{x}^3 \quad \dots$$

$$\vec{x}^{k_1} \quad \vec{x}^{k_2} \quad \vec{x}^{k_3} \quad \dots$$

$$\vec{x}^{k_{n_1}} \quad \vec{x}^{k_{n_2}} \quad \vec{x}^{k_{n_3}} \quad \dots$$

Hint: Remember a subsequence of a sequence x_n corresponds to a function $f: \mathbb{N} \to \mathbb{N}$ that is (1) injective and (2) increasing.

$$x_n \longrightarrow x_{f(n)} \text{ (usually write } x_{k_n}\text{)}$$

A subsequence of a subsequence corresponds to two such functions f and g. That is

$$x_n \longrightarrow x_{f(n)} \longrightarrow x_{g(f(n))} = x_{g\circ f(n)}$$

This is a way to understand passing to subsequences multiple times.

Often what people say is this: Since x_1^k is bounded, we can pass to a subsequence that converges, and hence define a subsequence of $\vec{x}^k$ such that its first component converges. Hence by passing to a subsequence, we can assume that the first component of $\vec{x}^k$ converges. We can do this over and over.

3. Remarks.

   a. We want to characterize the sets that have a property like that described in the B-W Theorem, specifically: *For which sets A is it true that any sequence $\{\mathbb{x}^k\} \subseteq A$ has a convergent subsequence?*

   b. B-W Theorem says that if A is a closed ball, this property holds.

   c. It is also possible to think of this property as saying: *Which subsets of $\mathbb{R}^n$ are most like finite sets in a particular sense?*

4. <u>Definition.</u> (8.3.1) An *open cover* of a subset $S \subseteq \mathbb{R}^n$ is a collection of open sets (possibly infinite) $\mathcal{O} = \{O_\alpha : \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} O_\alpha$. The set S is said to be *compact* if every open cover of S has a finite subcover, that is if $\{O_\alpha : \alpha \in A\}$ is an open cover of S then there exist indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that
$$S \subseteq \bigcup_{k=1}^{n} O_{\alpha_k}.$$

5. <u>Examples.</u> (a) An open ball $B(\mathbb{a}, r)$ is not compact.

   (b) A finite subset of $\mathbb{R}^n$ is compact.