Midterm 3-6

All ~~proofs~~

Learn the proofs of Theorems done in class or assigned as exercises in class. Also look at examples.

Looking at $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$.

$\mathbb{R}^n$ is a normed linear space.

$$\|\vec{x}\| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} = (\langle \vec{x}, \vec{x} \rangle)^{1/2}.$$

$\mathbb{R}^n$ inherits many of the topological properties of $\mathbb{R}$.

(a) $\mathbb{R}^n$ is complete. (Cauchy $\Rightarrow$ convergent)

(b) ~~Charact~~ Compactness — Heine-Borel property

(c) Bolzano-Weierstrass

(d) compactness $\Leftrightarrow$ H-B $\Leftrightarrow$ B-W $\Leftrightarrow$ closed + bounded

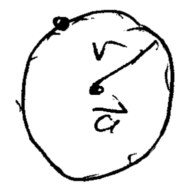
(e) $\mathbb{R}^n$ has a countable dense subset, $\mathbb{Q}^n$.

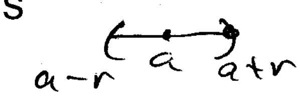
Important fact: $\vec{x}^k \to \vec{x}$ in $\mathbb{R}^n$ $\Leftrightarrow$ $x_j^k \to x_j$ for all $1 \le j \le n$.

8.2. Open Sets and Closed Sets.

A. Open Sets.

1. Definition. The *open ball centered at* $\mathbb{a} \in \mathbb{R}^n$ *with radius* $r > 0$, denoted $B(\mathbb{a}, r)$ is the set
$$B(\mathbb{a}, r) = \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x} - \mathbb{a}\| < r\}.$$
A set $\mathcal{O} \subseteq \mathbb{R}^n$ is *open* if for each $\mathbb{x} \in \mathcal{O}$, there is an $r > 0$ such that
$$B(\mathbb{x}, r) \subseteq \mathcal{O}$$

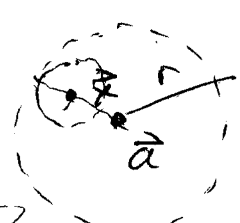




2. Examples.
   a. The empty set $\emptyset$ is open, and $\mathbb{R}^n$ is open.
   b. Any open ball is an open set.
   c. Any open set can be written as the union of a collection of open balls.

a. $\emptyset$ is open: $\forall \vec{x} \in \emptyset$, anything holds.
$\mathbb{R}^n$ is open: Let $\vec{x} \in \mathbb{R}^n$, let $r = 1$, $B(\vec{x}, 1) \subseteq \mathbb{R}^n$.

b. Let $B(\vec{a}, r)$ be given. Let $\vec{x} \in B(\vec{a}, r)$.
Find $\varepsilon > 0$ such that $B(\vec{x}, \varepsilon) \subseteq B(\vec{a}, r)$.



Let $\varepsilon = r - \|\vec{x} - \vec{a}\|$, and let $\vec{y} \in B(\vec{x}, \varepsilon)$.
Then $\|\vec{y} - \vec{a}\| \leq \|\vec{y} - \vec{x}\| + \|\vec{x} - \vec{a}\|$
$$< \varepsilon + \|\vec{x} - \vec{a}\| = r$$

c. Idea: $\mathcal{O} \subseteq \mathbb{R}^n$ open. ~~Let~~ $\forall \vec{x} \in \mathcal{O}$, $\exists \varepsilon_x > 0$ s.t. $B(\vec{x}, \varepsilon_x) \subseteq \mathcal{O}$. In fact $\mathcal{O} = \bigcup_{x \in \mathcal{O}} B(\vec{x}, \varepsilon_x)$.

3. <u>Theorem.</u> The union of any collection of open sets is open.

<u>Proof.</u> Let $\{\mathcal{O}_\alpha\}_{\alpha \in A}$, A some index set, be a collection of open sets. Show $\mathcal{O} = \bigcup_{\alpha \in A} \mathcal{O}_\alpha$ is open. Let $\vec{x} \in \mathcal{O}$. Then $\vec{x} \in \bigcup_{\alpha} \mathcal{O}_\alpha$ so there is a $\alpha_0 \in A$ such that $\vec{x} \in \mathcal{O}_{\alpha_0}$. Since $\mathcal{O}_{\alpha_0}$ is open there is an $r > 0$ such that $B(\vec{x}, r) \subseteq \mathcal{O}_{\alpha_0} \subseteq \mathcal{O}$. Hence $\mathcal{O}$ is open.

4. <u>Theorem.</u> The intersection of a finite number of open sets is open. The infinite intersection of open sets need not be open.

<u>Proof.</u> Let $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n\}$ be open sets. Let $\mathcal{O} = \bigcap_{j=1}^{n} \mathcal{O}_j$. Show $\mathcal{O}$ is open. Let $\vec{x} \in \mathcal{O}$. Then $\vec{x} \in \mathcal{O}_j$ for all j. Then there are $r_j > 0$ such that $B(\vec{x}, r_j) \subseteq \mathcal{O}_j$ all j. Let $r = \min_{1 \le j \le n} r_j$

then if $\vec{x} \in \mathcal{O}$, $B(\vec{x}, r) \subseteq B(\vec{x}, r_j) \subseteq \mathcal{O}_j$

Hence $B(\vec{x}, r) \subseteq \bigcap_{j=1}^{n} \mathcal{O}_j = \mathcal{O}$.

Let $\mathcal{O}_j = B(\vec{0}, \frac{1}{j})$. Then $\bigcap_{j=1}^{\infty} \mathcal{O}_j = \{\vec{0}\}$. which is not open.

B. Closed Sets.

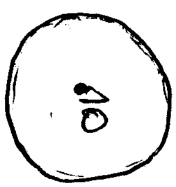
1. Definition. Let $A \subseteq \mathbb{R}^n$. The point $\mathbb{x}$ is a *limit point* of A if for every $\epsilon > 0$, there is a $\mathbb{y} \in A$ such that $0 < \|\mathbb{x} - \mathbb{y}\| < \epsilon$.

or a cluster point.

2. Remark.

a. A limit point of a set A need not be an element of A. For example, what are the limit points of $B(\mathbb{O}, 1)$?

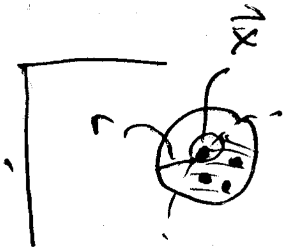
$\vec{x}$ is a limit point of $B(\vec{0},1)$ iff. $\|\vec{x}\| \le 1$

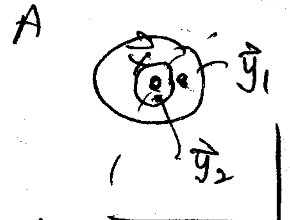


Pf Exercise

b. Claim. A point $\mathbb{x}$ is a limit point of a set A if and only if for every $r > 0$, $B(\mathbb{x}, r) \cap A$ contains infinitely many points.

Proof. ($\Longrightarrow$) Spse $\vec{x}$ is a limit pt of A, let $r > 0$. Then there is a $\vec{y} \in A$ with $0 < \|\vec{x} - \vec{y}\| < r$. Hence $\vec{y} \in B(\vec{x}, r) \cap A$. Next let $r_1 = \frac{\|\vec{x} - \vec{y}\|}{2} < \|\vec{x} - \vec{y}\| < r$. Since $\vec{x}$ is a limit pt of A, there is $\vec{y}_1 \in A$ with $0 < \|\vec{x} - \vec{y}_1\| < r_1$. Let $r_2 = \frac{\|\vec{x} - \vec{y}_1\|}{2}$ and continue in this fashion. We obtain a sequence $\vec{y}_k \in B(\vec{x}, r) \cap A$. I claim that $\vec{y}_k \neq \vec{y}_j$ for $k \neq j$. Look at $\|\vec{y}_k - \vec{y}_j\|$



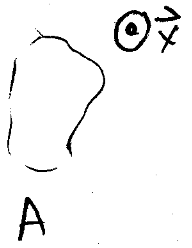


Assume $k > j$ $\|\vec{y}_k - \vec{y}_j\| \ge \|\vec{y}_j - \vec{x}\| - \|\vec{y}_k - \vec{x}\|$

But $\|y_k - \vec{x}\| < r_k < r_{k-1} < \cdots < r_j = \frac{\|\vec{x} - \vec{y}_j\|}{2}$

Hence $\|\vec{y}_k - \vec{y}_j\| \geq \|\vec{y}_j - \vec{x}\| - \frac{\|\vec{y}_j - \vec{x}\|}{2} = \frac{\|\vec{y}_j - \vec{x}\|}{2} > 0$

($\Leftarrow$) Exercise



c. Definition. Let $A \subseteq \mathbb{R}^n$. The point $\mathbb{x}$ is an *isolated point* of A if there exists an $r > 0$ such that $B(\mathbb{x}, r) \cap A = \cancel{\emptyset}\ \{\vec{x}\}$. Note that this implies that $\vec{x} \in A$.

d. Claim. Every point of a set $A \subseteq \mathbb{R}^n$ is either an isolated point of A or a limit point of A.

Pf: Exercise

3. Definition. A set $F \subseteq \mathbb{R}^n$ is said to be *closed* if it contains all of its limit points.

4. Examples.
   a. The empty set $\emptyset$ is closed, and $\mathbb{R}^n$ is closed. This also shows that it is possible for a set to be both open and closed.
   b. An open ball $B(\mathbb{x}, r)$ is not closed.
   c. Any *closed ball*

      $\overline{B(\mathbb{a}, r)} = \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x} - \mathbb{a}\| \le r\}$

      is closed.
   d. Any finite set is closed.

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5. Theorem. A set is closed if and only if its complement is open.

4. (a) $\vec{x}$ is a limit pt of $\emptyset$ means $\forall r>0$ there is $\vec{y}\in\emptyset$ with $0<\|\vec{x}-\vec{y}\|<r$.

Hence set of limit pts of $\emptyset$ is $\emptyset$.

(b) Look at $B(\vec{x},r)$.
What are limit pts of $B(\vec{x},\vec{r})$?

$B(\vec{x},r)$

Ans: $\{\vec{y} : \|\vec{x}-\vec{y}\|\le r\}$

Hence $B(\vec{x},r)$ does not contain all its limit pts.