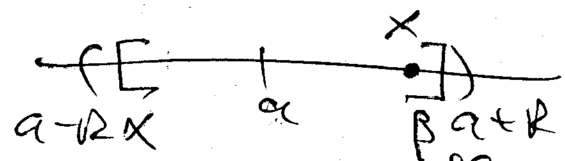
Spse $f(x)=\sum_{k=0}^{\infty} c_k(x-a)^k$ with radius of convergence $R>0$. Then $f'(x)$ exists for all $|x-a|<R$ and

$$f'(x)=\sum_{k=0}^{\infty} k\, c_k (x-a)^{k-1}$$ has radius of convergence R.

Given x with $|x-a|<R$, $\exists\ [\alpha,\beta] \subseteq (a-R, a+R)$ such that $x \in [\alpha,\beta]$.



Then show $\sum_{0}^{\infty} c_k(x-a)^k$ conv abs & unif on $[\alpha,\beta]$. Apply Thm 5.5.1 for the rest of it.

Claim: $\sum_{k=0}^{\infty} k c_k (x-a)^k$ has R.O.C. R.

Pf: Show $\limsup_k |k c_k|^{1/k} = \limsup_k |c_k|^{1/k}$

Informally $\limsup_k (k)^{1/k} |c_k|^{1/k}$

$=(\limsup k^{1/k})(\limsup |c_k|^{1/k})$ BUT NOT TRUE.

~~ex~~ $\limsup_k (x_k y_k) \leq (\limsup x_k)(\limsup y_k)$

e.g. $x_k=(-1)^k$, $y_k=(-1)^{k+1}$

$x_k y_k = -1$ so $\limsup (x_k y_k) = -1$

but $\limsup x_k = 1 = \limsup y_k$

What is true is this.

If $x_k \to x$ then

$$\limsup_k (x_k y_k) = (\lim x_k)(\limsup y_k)$$

(Exercise)

Hint: Already know

$$\limsup (x_k y_k) \le (\limsup x_k)(\limsup y_k) = (\lim x_k)(\limsup y_k)$$

Show: For every $\varepsilon > 0$,

$$\limsup (x_k y_k) > (\lim x_k)(\limsup y_k) - \varepsilon$$

(same as: $\limsup x_k y_k \ge (\lim x_k)(\limsup y_k)$)

Therefore: $\limsup (k^{1/k} |c_k|^{1/k})$

$$= \underbrace{(\lim_k k^{1/k})}_{1} (\limsup_k |c_k|^{1/k})$$

8.1. Euclidean Space.

<u>Definition.</u> Let n be a natural number. The set $\mathbb{R}^n$, defined by

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \dots, x_n) : x_j \in \mathbb{R}\}$$

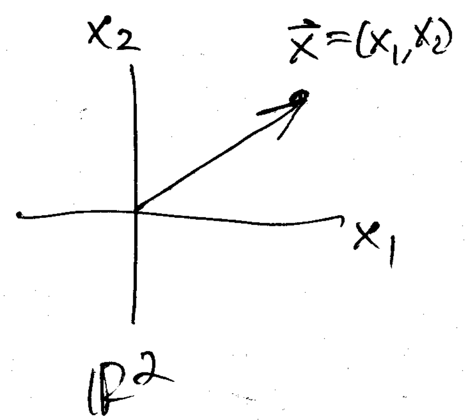
Is the Cartesian product of n copies of $\mathbb{R}$. We usually write $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

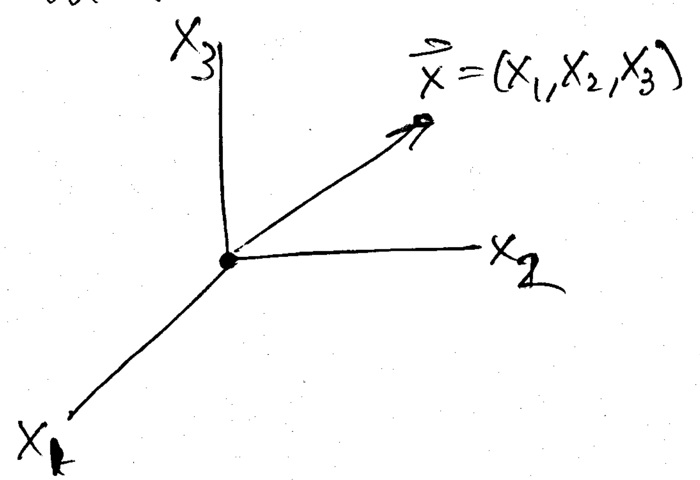
<u>Remark</u>. (a) $\mathbb{R}^2$ is the Cartesian plane and $\mathbb{R}^3$ is Cartesian 3-space. We say $\mathbb{R}^n$ is <u>Euclidean</u> n-space.
(b) $\mathbf{x} = (x_1, x_2, \dots, x_n) = \mathbf{y} = (y_1, y_2, \dots, y_n)$ if and only if $x_j = y_j$ for all j. The vector $\mathbf{0} = (0, 0, \dots, 0)$ is the zero vector or the origin.
(c) So far, $\mathbb{R}^n$ has been defined only as a set, but other structure can be imposed on it.

A. Algebraic structure

$\mathbb{R}^n$ is a *vector space* (see the definition and axioms on p. 59).

We will use $\mathbb{R}^2$ and $\mathbb{R}^3$ for sketches and motivation and illustration.





B. Geometric Structure

1. Definition. The *dot product* (or *scalar product,* or *inner product*) of $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$, denoted $\mathbb{x} \cdot \mathbb{y}$ or $\langle \mathbb{x}, \mathbb{y} \rangle$ is given by

$$\mathbb{x} \cdot \mathbb{y} = \langle \mathbb{x}, \mathbb{y} \rangle = \sum_{j=1}^{n} x_j y_j = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

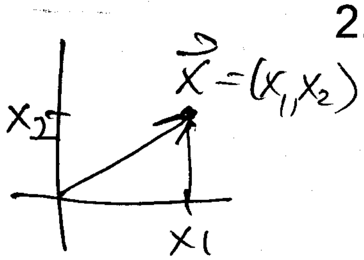
2. The interaction of the algebraic and geometric structure of $\mathbb{R}^n$ is given in Definition 8.1.1 in the book. This definition also gives the defining characteristics of a scalar product.

3. The inner product defines a geometric structure on $\mathbb{R}^n$ because it allows us to define a notion of the *angle between* $\mathbb{x}$ *and* $\mathbb{y}$*.* More on this later.

C. Topological Structure

1. Definition. The *(Euclidean) norm* of $\mathbb{x} \in \mathbb{R}^n$, denoted $\|\mathbb{x}\|$ or sometimes $\|\mathbb{x}\|_2$ is

$$\|\mathbb{x}\| = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} = (\langle \mathbb{x}, \mathbb{x} \rangle)^{1/2}$$



2. Remark. (a) $\|\mathbb{x}\|$ is the usual notion of the length of the arrow representing the vector $\mathbb{x} \in \mathbb{R}^2$ or $\mathbb{R}^3$ and generalizes the notion of absolute value on $\mathbb{R}$.

$\|x\| = (x^2)^{1/2} = |x|$

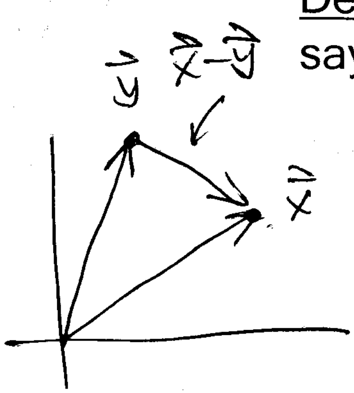
$\|\vec{x}\|$ = length of arrow (Pythag Thm)

(b) The norm defines a notion of *distance* by denoting the distance between $\mathbb{x}$ and $\mathbb{y}$ as $\|\mathbb{x} - \mathbb{y}\| = \|\mathbb{y} - \mathbb{x}\|$.

$\|\vec{x}\| = (x_1^2 + x_2^2)^{1/2}$

3. Now that we have a notion of distance in $\mathbb{R}^n$, we can talk about convergence of sequences, viz.

Definition. Let $\mathbb{x}^{(k)}$ be a sequence in $\mathbb{R}^n$. We say that $\mathbb{x}^{(k)} \to \mathbb{x}$ if $\left\|\mathbb{x}^{(k)} - \mathbb{x}\right\| \to 0$ as $k \to \infty$.



$\vec{x}^{(k)} = (x_1^{(k)}, x_2^{k}, x_3^{k}, \ldots, x_n^{k})$

4. Theorem.

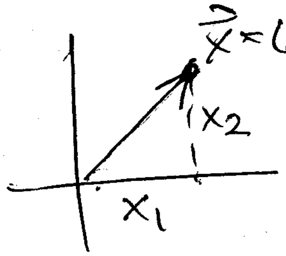
A sequence $\mathbb{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ in $\mathbb{R}^n$ converges to $\mathbb{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ if and only if for each j, $\lim_{k\to\infty} x_j^{(k)} = x_j$.

Proof:

($\Rightarrow$) Suppose $\vec{x}^k \to \vec{x}$ in $\mathbb{R}^n$. This means $\|\vec{x}^k - \vec{x}\| \to 0$ as $k \to \infty$. But given $1 \le j \le n$,

$$|x_j^k - x_j|^2 \le \sum_{l=1}^{n} |x_l^k - x_l|^2 = \|\vec{x}^k - \vec{x}\|^2$$

Therefore for each j, $|x_j^k - x_j| \to 0$ as $k \to \infty$.



$|x_1| \le \|\vec{x}\|$

$|x_2| \le \|\vec{x}\|$

($\Leftarrow$) Suppose $x_j^k \to x_j$ for each $1 \le j \le n$. Must show that $\|\vec{x}^k - \vec{x}\| \to 0$ as $k \to \infty$.

Idea: $\|\vec{x}^k - \vec{x}\|^2 = \sum_{j=1}^{n} |x_j^k - x_j|^2$

Let $\varepsilon > 0$, choose N_1 so that $k \ge N_1$ implies $|x_1^k - x_1| < \frac{\varepsilon}{\sqrt{n}}$, choose N_2 so that $k \ge N_2$ implies $|x_2^k - x_2| < \frac{\varepsilon}{\sqrt{n}}$, and in general N_j so that $k \ge N_j$ implies $|x_j^k - x_j| < \frac{\varepsilon}{\sqrt{n}}$.

Let $N = \max_{1 \le j \le n} N_j$. If $k \ge N$ then

$$\|\vec{x}^k - \vec{x}\| = \left(\sum_{j=1}^{n} |x_j^k - x_j|^2\right)^{1/2} < \left(\sum_{j=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right)^{1/2} = \varepsilon\left(\sum_{j=1}^{n} \frac{1}{n}\right)^{1/2} = \varepsilon.$$

Say

$$\|\vec{x}^k - \vec{x}\| = \left(\sum_{j=1}^{n} |x_j^k - x_j|^2\right)^{1/2}$$

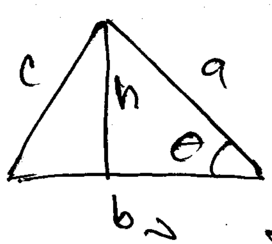
$$\le \max_{1\le j\le n} |x_j^k - x_j| \left(\sum_{j=1}^{n} 1\right)^{1/2}$$

$$= \sqrt{n}\ \underbrace{\max_{1\le j\le n} |x_j^k - x_j|}$$

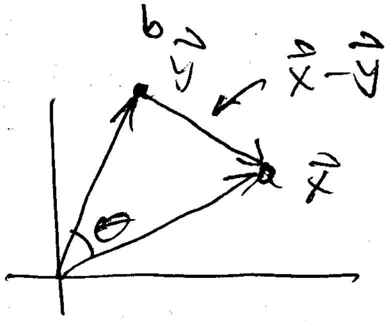
D. Interaction of topological and geometric structure.

1. <u>Claim</u>. $||\mathbb{x} - \mathbb{y}||^2 = ||\mathbb{x}||^2 + ||\mathbb{y}||^2 - 2\langle \mathbb{x}, \mathbb{y}\rangle$

$$||\vec{x}-\vec{y}||^2 = \langle \vec{x}-\vec{y}, \vec{x}-\vec{y}\rangle$$
$$= \langle \vec{x},\vec{x}\rangle - \langle \vec{x},\vec{y}\rangle - \langle \vec{y},\vec{x}\rangle + \langle \vec{y},\vec{y}\rangle$$
$$= ||\vec{x}||^2 + ||\vec{y}||^2 - 2\langle \vec{x},\vec{y}\rangle$$



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



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2. If $\mathbb{x}, \mathbb{y} \in \mathbb{R}^2$ then the Law of Cosines says that
$$||\mathbb{x} - \mathbb{y}||^2 = ||\mathbb{x}||^2 + ||\mathbb{y}||^2 - 2||\mathbb{x}||\,||\mathbb{y}||\cos\theta$$
where θ is the angle between $\mathbb{x}$ and $\mathbb{y}$.

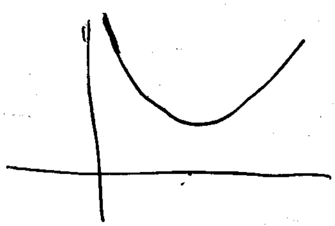
3. This implies that $\langle \mathbb{x}, \mathbb{y}\rangle = ||\mathbb{x}||\,||\mathbb{y}||\cos\theta$ and therefore we can *define* the angle between any two vectors in $\mathbb{R}^n$ in this way.

4. <u>Theorem</u>. (Cauchy-Schwarz inequality)
Given $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$, $|\langle \mathbb{x}, \mathbb{y} \rangle| \leq \|\mathbb{x}\| \|\mathbb{y}\|$ with equality holding if and only if $\mathbb{x}$ and $\mathbb{y}$ are parallel, that is, one is a scalar multiple of the other.

Proof: Let $t \in \mathbb{R}$. Look at $\|\vec{x} - t\vec{y}\|^2$

Then $0 \leq \|\vec{x} - t\vec{y}\|^2 = \|\vec{x}\|^2 + t^2\|\vec{y}\|^2 - 2t\langle \vec{x}, \vec{y} \rangle$.

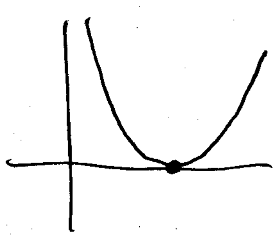
This is a quadratic polynomial in t which is non-negative, so discriminant is ≤ 0. That is,



$$(-2\langle \vec{x}, \vec{y} \rangle)^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \leq 0$$

But this is exactly $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

Note that if $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|$ then the disc. of the quadratic polynomial is zero.



This means that there is a $t_0 \in \mathbb{R}$ such that $\|\vec{x} - t_0\vec{y}\|^2 = 0$, i.e. $\vec{x} = t_0\vec{y}$.

The other implication: if $\vec{x} = t_0\vec{y}$ or $\vec{y} = t_0\vec{x}$, then $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|$ is an exercise.

5. Theorem. Let $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$. Then

a. $\|\mathbb{x}\| \geq 0$ with equality holding if and only if $\mathbb{x} = \mathbb{0}$.

b. $\|\alpha\mathbb{x}\| = |\alpha|\|\mathbb{x}\|$ for all $\alpha \in \mathbb{R}$.

c. $\|\mathbb{x} + \mathbb{y}\| \leq \|\mathbb{x}\| + \|\mathbb{y}\|$. (Triangle inequality)

Proof: (a) and (b) are exercises.

To prove (c) let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\langle x, y\rangle$$

$$\leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2|\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\|\vec{y}\|$$

$$= (\|\vec{x}\| + \|\vec{y}\|)^2$$

In fact any function $\|\cdot\| : V \to \mathbb{R}$ satisfying (a)–(c) is a norm on V. In fact

$$\|\vec{x}\|_\infty = \max_{1 \leq j \leq n} |x_j| \text{ is also a norm,}$$

$$\|\vec{x}\|_1 = \sum_{j=1}^{n} |x_j| \text{ is a norm, and if } p > 1$$

$$\|\vec{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \text{ is a norm.}$$