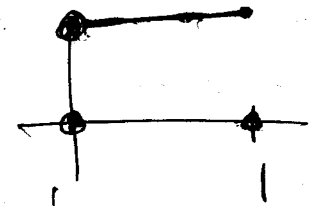
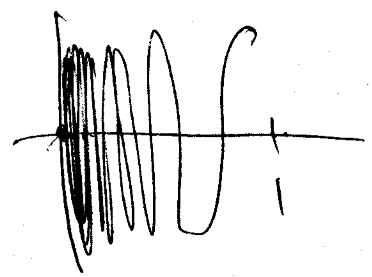
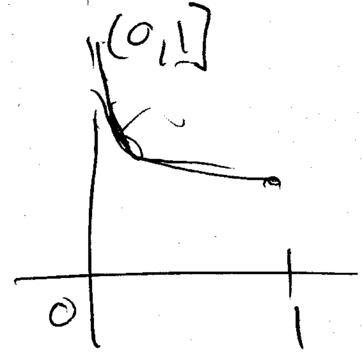
5.22 Know $|f'(x)| \le 1$ on $(0,1]$.

(a) Prove $\lim_{n\to\infty} f(\frac{1}{n})$ exists.





$\sin(\frac{1}{x})$ cont on $(0,1]$



$\frac{1}{x}$ cont on $(0,1]$

can't be extended continuously

Idea: $\sum_{n=1}^{\infty} \left|f(\frac{1}{n+1}) - f(\frac{1}{n})\right|$ converges.

$$\left|f(\tfrac{1}{n+1}) - f(\tfrac{1}{n})\right| \le |f'(\mu)| \left|\tfrac{1}{n+1} - \tfrac{1}{n}\right| \le \frac{1}{n(n+1)} \le \frac{1}{n^2}$$

↑ summable by comp test.

↑ summable

Abs conv of $\sum \left|f(\frac{1}{n+1}) - f(\frac{1}{n})\right| \Longrightarrow$

conv of $\sum_{n=1}^{\infty} \left(f(\frac{1}{n+1}) - f(\frac{1}{n})\right)$ ← telescoping

$$S_n = \sum_{k=1} f(\tfrac{1}{k+1}) - f(\tfrac{1}{k}) = f(\tfrac{1}{2}) - f(1) + \cancel{f(\tfrac{1}{3})} - \cancel{f(\tfrac{1}{2})} + \cdots + f(\tfrac{1}{n+1}) - \cancel{f(\tfrac{1}{n})}$$

$= f(\frac{1}{n+1}) - f(1)$ Know: $S_n \to S$, so

$f(\frac{1}{n+1}) \to \underbrace{f(1)+S}_{L}$.

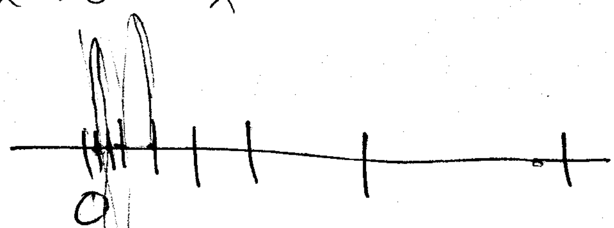
b) Prove $\lim_{x \to 0^+} f(x) = L$

Know $\boxed{\lim_{n \to \infty} f(\frac{1}{n}) = L}$.

$x \to 0^+$ $\quad$ $x$

Sequential criterion: $\lim_{x \to a} f(x) = L \iff \forall$ sequences $x_n \to a$, $f(x_n) \to L$.

Idea:



$\lim_{x \to 0^+} f(x) = L$ means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in (0, \delta)$ then $|f(x) - L| < \varepsilon$.

Plan: Look at $|f(x) - L| \leq |f(x) - f(\frac{1}{n})| + |f(\frac{1}{n}) - L|$

$\leq |x - \frac{1}{n}| + |f(\frac{1}{n}) - L|$

etc ---

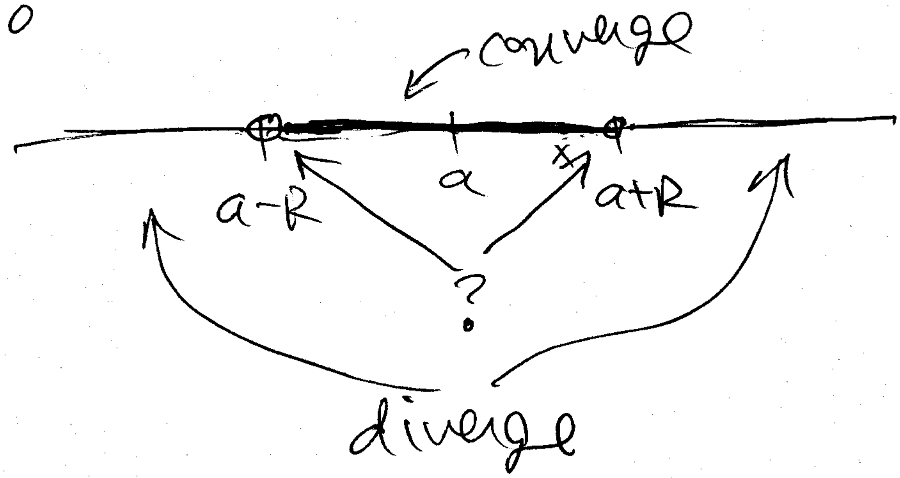
$x_n \to x$. This means $|x_n - x| < \cancel{\varepsilon}$

This means $\forall \varepsilon > 0$, $|x_n - x| < \cancel{\varepsilon}$

This means $\forall \varepsilon > 0$ $\exists N$ such that

if $n \geq N$ then $|x_n - x| < \varepsilon$.

$\sum_{0}^{\infty} c_k (x-a)^k \quad \exists R > 0$ s.t.




What is $R$? $\quad \frac{1}{R} = \limsup_{k} (|c_k|)^{1/k}$.

can have $R=0$ or $R=\infty$.

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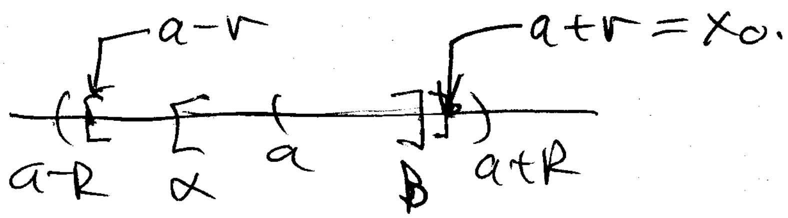
Claim: Let $\sum_{0}^{\infty} c_k (x-a)^k$ have R.O.C. $R > 0$.

Let $[\alpha, \beta] \subseteq (a-R, a+R)$. Then series conv. abs. and uniformly on $[\alpha, \beta]$.

Pf: Idea: use Weierstrass M-test.




Since $[\alpha, \beta] \subseteq (a-R, a+R)$, then there is $0 < r < R$ such that $a-R < a-r < \alpha \le \beta < a+r < a+R$. So given $x \in [\alpha, \beta]$, $|x-a| < r < R$ and also

$$\sup_{x \in [\alpha, \beta]} |c_k| |x-a|^k \le |c_k| r^k < |c_k| R^k.$$

If we let $x_0 = a + r$ then $|x_0 - a| = r$ and we know $\sum_{k=0}^{\infty} c_k (x_0 - a)^k = \sum_{k=0}^{\infty} c_k r^k$ conv. absolutely

Hence by M-test, $\sum_{0}^{\infty} c_k (x-a)^k$ conv absolutely and uniformly on $[\alpha, \beta]$.

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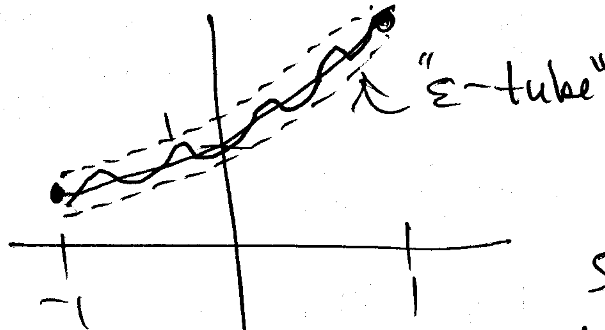
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{conv unif on } [-1,1]$$




$\sum_{k=0}^{n} \frac{x^k}{k!}$ n large.

$$\sup_{x \in [-1,1]} \left| \sum_{0}^{n} \frac{x^k}{k!} - e^x \right| < \varepsilon$$

if n large enough.

Theorem. (Abel) Let $\alpha < \beta$. If the power series $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ converges pointwise on $[\alpha, \beta]$, then $f(x)$ is continuous and converges uniformly on $[\alpha, \beta]$.

Proof. Without loss of generality let $[\alpha, \beta] = [a, a+R]$. Let $x \in [a, a+R]$, $x \neq a, a+R$

Look at $c_k (x-a)^k = c_k R^k \left(\frac{x-a}{R}\right)^k$

I know: $\sum_{k=0}^{\infty} c_k R^k$ converges (by hypothesis)

and I know $\left(\frac{x-a}{R}\right)^k \searrow 0$ since $x-a \leq R$.

By summation by parts

$$\sum_{k=m}^{n} c_k (x-a)^k = \sum_{k=m}^{n} c_k R^k \left(\frac{x-a}{R}\right)^k$$

$$= \sum_{k=m}^{n} c_k R^k \underbrace{\left[\frac{x-\cancel{a}}{R}\right]^n}_{\leq 1} - \sum_{k=m}^{n-1} \left(\sum_{j=m}^{k} a_j R^j\right) \left(\left(\frac{x-a}{R}\right)^{k+1} - \left(\frac{x-a}{R}\right)^k\right)$$

$$\left|\sum_{k=m}^{n} c_k (x-a)^k\right| \leq \left|\sum_{k=m}^{n} c_k R^k\right| + \max_{m \leq k \leq n-1} \left|\sum_{j=m}^{k} a_j R^j\right| \left(\left(\frac{x-a}{R}\right)^n - \left(\frac{x-a}{R}\right)^m\right)$$

Since $\left|\frac{x-a}{R}\right| \le 1$, for all $n, m$,

$$\left|\left(\frac{x-a}{R}\right)^n - \left(\frac{x-a}{R}\right)^m\right| \le 2 \text{ indep of } x.$$

Since $\sum_{k=0}^{\infty} c_k R^k$ converges, given $\varepsilon > 0$ there is an $N$ such that if $n, m \ge N$ then $\left|\sum_{k=m}^{n-1} c_k R^k\right| < \frac{\varepsilon}{3}$

Hence also, $\max_{m \le k \le n-1} \left|\sum_{j=m}^{k} c_j R^j\right| < \frac{\varepsilon}{3}$.

Finally then -

$$\left|\sum_{k=m}^{n} c_k (x-a)^k\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}(2) = \varepsilon.$$

Since $N$ is indep of $x$, convergence is uniform.

Theorem. If $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ has radius of convergence $R > 0$, then $f$ is infinitely differentiable on $(a-R, a+R)$, and $f^{(n)}$ is given by

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)c_k(x-a)^{k-n}$$

which power series also has radius of convergence $R$.

Proof: Idea: Look at $n=1$.

We would like to say, $f(x) = \sum_0^{\infty} c_k (x-a)^k$, that $f'(x) = \sum_{k=0}^{\infty} k c_k (x-a)^{k-1}$. How can we verify we can do this? First of all take $[\alpha, \beta] \subseteq (a-R, a+R)$. Use Thm 5.5.1 on this interval. Part we need to verify is that $\sum_{k=0}^{\infty} k c_k (x-a)^{k-1} = (x-a)^{-1} \sum_{k=0}^{\infty} k c_k (x-a)^k$ conv uniformly on $[\alpha, \beta]$.

Idea: Show that $\limsup (k|c_k|)^{1/k} = \frac{1}{R}$

$= \limsup_k (|c_k|)^{1/k}$.