<u>Definition.</u> (Iterated Sums) Let $a_{j,k}$ be a doubly indexed sequence. The *iterated sum* of the sequence is defined as the series.

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)$$

In other words, if we define the sequence c_k by $c_k = \sum_{j=1}^{\infty} a_{j,k}$ then

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} c_k$$

<u>Remark</u> Note that the iterated sum is <u>not</u> a rearrangement of the series $\sum_{j,k=1}^{\infty} a_{j,k}$.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & - & - \\ & \vdots & & & \end{array}$$

Theorem. If $a_{j,k}$ is absolutely summable then

$$\sum_{j,k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k}$$

Proof. Outline form.

(1) First show c_k exists for all k.

$c_k = \sum_{j=1}^{\infty} a_{j,k}$. ~~Use Cauchy criterion~~

Use ~~and~~ fact that $\sum_{j,k=1}^{\infty} |a_{j,k}| < \infty$. Enough to show $\sum_{j=1}^{\infty} |a_{j,k}|$ by showing partial sums bounded.

$$\sum_{j=1}^{n} |a_{j,k}| \le \sum_{j=1}^{n} \sum_{p=1}^{n} |a_{j,p}| \quad \text{if } n \ge k.$$

$$\begin{array}{cccc} & & k=3 \\ & & \downarrow \\ a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

This is bounded sequence b/c. $\sum_{j,k} |a_{j,k}|$ converges.

(2) Show $\sum_{k=1}^{\infty} c_k$ converges. Enough to show $\sum_{k=1}^{\infty} |c_k|$ converges, i.e. that partial sums bounded.

$$\sum_{k=1}^{n} |c_k| = \sum_{k=1}^{n} \sum_{j=1}^{\infty} |a_{j,k}|$$

(3) Show $\sum_{k=1}^{\infty} c_k = \sum_{j,k=1}^{\infty} a_{j,k}$.

Start with

$$\left| \sum_{k=1}^{n} c_k - \sum_{k=1}^{n} \sum_{j=1}^{n} a_{j,k} \right|$$ show goes to 0.

Rest is an exercise

5.5/5.6. The Weierstrass M-test and Power Series.

~~∀ε>0 ∃K s.t. k≥K~~

Definition. A sequence of functions $f_k(x)$ defined on a domain D converges to the function $f(x)$ pointwise on D if for each $x \in D$ the numerical sequence $f_k(x)$ converges to $f(x)$. The convergence is *uniform* if

$$sup_{x\in D}\,|f_k(x) - f(x)| \to 0$$

As $k \to \infty$. The series $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise (resp. uniformly) on D if the sequence of partial sums $s_n(x) = \sum_{k=1}^{n} f_k(x)$ converges pointwise (resp. uniformly) on D.

$\forall \varepsilon > 0, x \in D,$ $\exists K$ such that $k \geq K$ then $|f_k(x) - f(x)| < \varepsilon$. (K may depend on x)

$\forall \varepsilon > 0, \exists K$ s.t. if $k \geq K$ then $\forall x \in D$, $|f_k(x) - f(x)| < \varepsilon$. (same K for all x)

Theorem 5.5.2. (Weierstrass M-test)
Let $f_k(x)$ be a sequence of functions defined on a domain D. Let

$$M_n = sup_{x\in D}\,|f_n(x)| = \|f_n\|_{sup} < \infty$$

If $\sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on D.

Proof: We must show that $\sum_{k=1}^{\infty} |f_k(x)|$ converges ~~uniformly~~ on D. This will show absolute convergence. For this, show that the partial sums $|s_n(x)| = |\sum_{k=1} f_k(x)|$ are bounded. This follows from the fact that $|s_n(x)| = |\sum_{k=1} f_k(x)| \leq \sum_{k=1} |f_k(x)| \leq \sum_{k=1} M_k$

$$\leq \sum_{k=1}^{\infty} M_k.$$

To show uniform convergence, consider $\sum_{k=m}^{n} f_k(x)$. If we can show that $\{S_n(x)\}$ is uniformly Cauchy, we will be done.

Def: Seq of numbers: x_n
$\forall \varepsilon>0 \ \exists N$ s.t. $n,m \geq N \Rightarrow |x_n - x_m| < \varepsilon$.

Seq of functions $g_n(x)$
$\forall \varepsilon>0 \ \exists N$ s.t. $n,m \geq N \Rightarrow \forall x, \ |g_n(x) - g_m(x)| < \varepsilon$
(uniformly Cauchy)
or $\forall \varepsilon>0 \ \exists N$ s.t. $n,m \geq N \Rightarrow \sup_x |g_n(x) - g_m(x)| < \varepsilon$

Let $\varepsilon>0$, find N so large that if $n,m \geq N$ then $\sum_{k=m}^{n-1} M_k < \varepsilon$. Then if $n,m \geq N$,

$$\left|\sum_{k=m}^{n-1} f_k(x)\right| \leq \sum_{k=m}^{n-1} |f_k(x)| \leq \sum_{k=m}^{n-1} M_k < \varepsilon.$$

So $S_n(x)$ is unif Cauchy since N independent of x.

<u>Definition</u> (Power Series)
A *power* series centered at a (or with *base* point a) is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

where c_k is a sequence of real coefficients.

<u>Theorem</u> 5.6.1. Given a power series $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, there exists a number $0 \le R \le \infty$, called the *radius of convergence* with the property that the series converges absolutely on the interval $|x-a| < R$, absolutely and uniformly on any interval $[\alpha, \beta] \subseteq (a-R, a+R)$, and diverges for $|x-a| > R$.

<u>Proof:</u> First of all, what is R?

$R = \dfrac{1}{\limsup_k \left(|c_k|^{1/k}\right)}$. If $|x-a| < R$ then

series is $\sum_{k=0}^{\infty} c_k (x-a)^k$ and by root test

we consider $\limsup_k \left(|c_k| |x-a|^k\right)^{1/k}$

Since $|x-a| < R$, $\limsup\left(|c_k| |x-a|^k\right)^{1/k}$

$< \limsup\left(|c_k| R^k\right)^{1/k} = R \underbrace{\limsup |c_k|^{1/k}}_{\le 1}$

and series converges abs. ~~absolutely~~.

e.g. $\sum_{k=0}^{\infty} c_k (x-a)^k$

$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

Partial sums are polynomials.

$c_k = \frac{1}{k}$, $a = 1 \to \sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$

$(c_0 = 0)$

Radius of convergence:

Ratio test $\left| \frac{\frac{(x-1)^{k+1}}{k+1}}{\frac{(x-1)^k}{k}} \right| = \frac{|x-1|}{1} \cdot \frac{k}{k+1} \to |x-1|$

$\therefore$ $|x-1| < 1$ convergence

$|x-1| > 1$ divergence

$|x-1| = 1$ not clear $\to$ $|x-1| = 1 \Rightarrow$

$x = 2$ or $x = 0$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

Interval of convergence is $[0, 2)$.

In this case, $\underline{R = 1}$

e.g. $R = \infty$ (take $a = 0$)

$\sum_{k=0}^{\infty} \frac{x^k}{k^2}$? $R = 1$ but converges on $[-1, 1]$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \left(= e^x\right) \qquad \left| \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} \right| = |x| \frac{k!}{(k+1)!} = |x| \frac{1}{k+1} \to 0$$

e.g. $R = 0$ (take $a = 0$)

$\sum_{k=0}^{\infty} k!\, x^k$ Ratio test gives ∞.

Similarly, if $|x-a| > R$ then
$\limsup (|c_k| |x-a|^k)^{1/k} > 1$ so series
diverges.