5.24 Suppose $\sum_{n=1}^{\infty} x_n$ converges conditionally

This means: ① $\sum_{n=1}^{\infty} x_n$ converges.

② $\sum_{n=1}^{\infty} |x_n|$ diverges

③ Extract 2 subsequences of x_n.

p_k subsequence of ≥ 0 terms

n_k subsequence of $<$ terms.

④ Know $x_n \to 0$ so also $p_k \to 0$, $n_k \to 0$.

y_k subseq of x_n means $y_k = x_{f(k)}$ where $f: \mathbb{N} \to \mathbb{N}$ is one-to-one

p_k, n_k not the same as x_n^+ and x_n^-.

Know: $\sum_{k=1}^{\infty} p_k = \infty$ and $\sum_{k=1}^{\infty} [-(n_k)] = \infty$.

Need to find a rearrangement of x_n, y_n, such that $\underbrace{\sum_{n=1}^{\infty} y_n = -\infty}_{*}$.

$*$ Given $M \in \mathbb{R}$ (think of $M<0$) there is an N such that for all $n \geq N$,

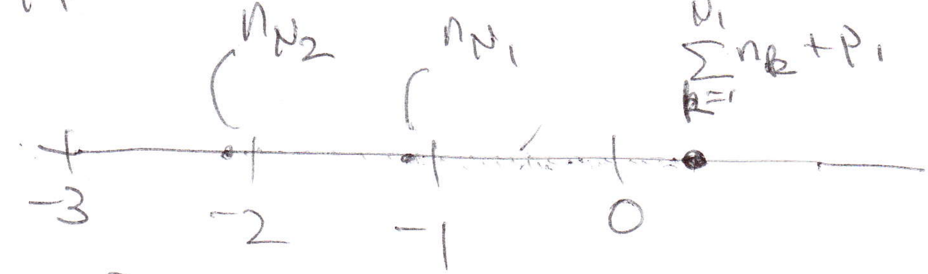
$S_n = \sum_{k=1}^{n} y_k < M$.

Idea: First let $M=-1$

Add up $n_1+n_2+n_3+\cdots+n_{N_1}$ until $\text{sum}_{N_1} < -1$

and N_1 is the first such index.

Then add p_1.



Then let $M=-2$

Add up $n_1+n_2+\cdots+n_{N_1}+p_1+n_{N_1+1}+\cdots+n_{N_2}$

until sum is <-2

Then add p_2. Continue like this

$\curvearrowright x_n$

y: $n_1, n_2, \ldots n_{N_1}, p_1, n_{N_1+1} \ldots n_{N_2}, p_2,$

$\ldots n_{N_m} \ldots n_{N_{m+1}}\ p_{m+1} \ldots$

Verify: a) This is a rearrangement.

b) $\displaystyle\sum_{n=1}^{\infty} y_n = -\infty$.

$f: \mathbb{N} \to \mathbb{N}$
one-to-one
onto.

Absolute convergence: $\sum_{n=1}^{\infty} x_n$, $x_n \geq 0$.

Ratio test:

① $\limsup_k \frac{x_{k+1}}{x_k} < 1$, converges

② $\liminf_k \frac{x_{k+1}}{x_k} \geq 1$, ~~conver~~ diverges

e.g. $1, 2, \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{8}, \ldots$

series converges.

but $\frac{x_{k+1}}{x_k} = \begin{cases} 2 & k \text{ odd} \\ \frac{1}{4} & k \text{ even} \end{cases}$

So $\limsup \frac{x_{k+1}}{x_k} = 2 \geq 1$

Root test works better.

2 series $\sum_{j=1}^{\infty} x_j$ $\sum_{k=1}^{\infty} y_k$ (convergent)

We can look at $\left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)$

We know:

$(x_1+x_2+x_3+x_4+x_5)(y_1+y_2+y_3+y_4)$

$=$ 20 terms but a single sum.

Think about $\{x_j y_k\}_{j,k=1}^{\infty}$.

	x_1	x_2	x_3	x_4	x_5
y_1	x_1y_1	x_2y_1	x_3y_1		
y_2	x_1y_2	x_2y_2	x_3y_2	$\cdots$	
y_3	x_1y_3	x_2y_3	x_3y_3		
y_4		$\vdots$			
y_5					

there are countably many terms So I can enumerate them

Can write as one sequence $\{z_n\}_{n=1}^{\infty}$

Issue is there are many ways to do this.

How do you make sense of $\sum_{j,k=1}^{\infty} x_j y_k$?

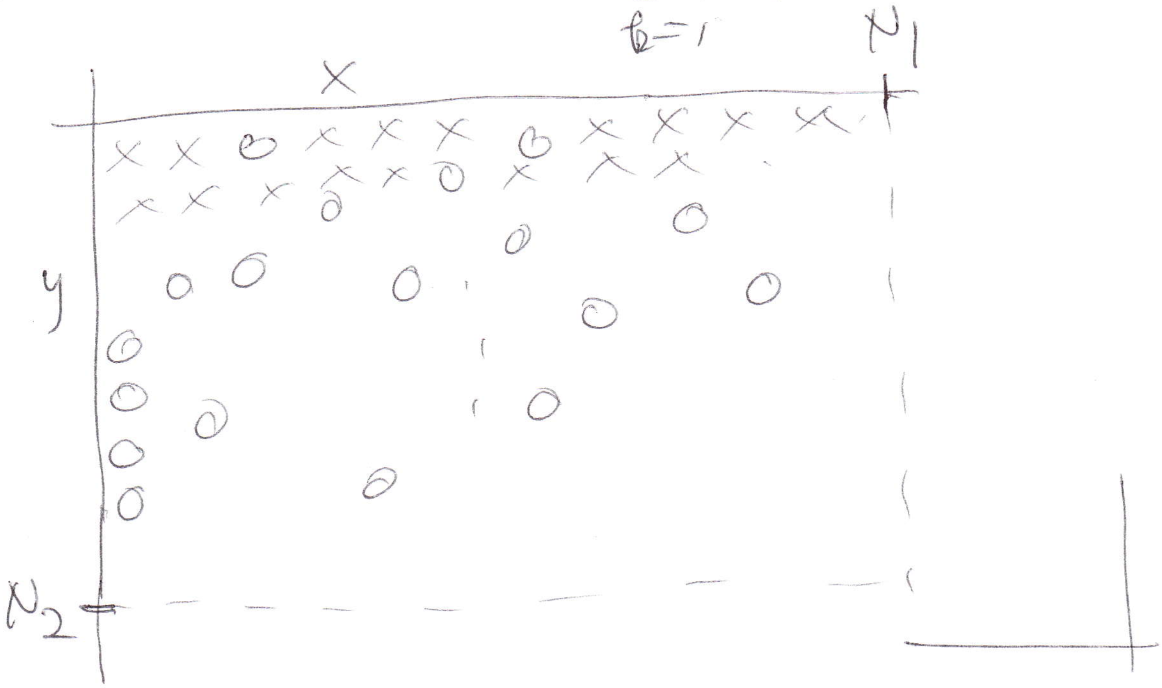
5.3. Products of Series.

<u>Theorem 5.3.2.</u> If x_n and y_n are absolutely summable, then so is the doubly indexed sequence $\{x_j y_k\}_{j,k=1}^{\infty}$ and

$$\sum_{j,k=1}^{\infty} x_j y_k = \left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)$$

<u>Proof:</u> Let z_n be any arrangement of $\{x_j y_k\}$. Must show $\sum_{n=1}^{\infty} z_n$ converges and equals $\left(\sum_1^{\infty} x_j\right)\left(\sum_1^{\infty} y_k\right)$. Will show that $\sum_{n=1}^{\infty} |z_n|$ converges by showing sequence of partial sums $s_n = \sum_{k=1}^{n} |z_k|$ is bounded.

<u>Idea:</u> Think about $\sum_{k=1}^{n} |z_k|$



Let $n \in \mathbb{N}$ and consider $S_n = \sum_{k=1}^{n} |z_k|$

Since $z_k = x_{f(k)} y_{g(k)}$ some $f, g : \mathbb{N} \to \mathbb{N}$.

let $N_1 = \max\{f(k) : 1 \leq k \leq n\}$ and

$N_2 = \max\{g(k) : 1 \leq k \leq n\}$ and let

$N_0 = \max(N_1, N_2)$. Then

$$S_n = \sum_{k=1}^{n} |z_k| \leq \sum_{k=1}^{N_0} |\overbrace{x_{f(k)} y_{g(k)}}^{|z_k|}| \leq \left(\sum_{j=1}^{N_0} |x_j|\right)\left(\sum_{k=1}^{N_0} |y_k|\right)$$

But since x_j and y_k are abs summable there is a bound M such that

$\sum_{j=1}^{N_0} |x_j|$ and $\sum_{k=1}^{N_0} |y_k| \leq M$ for all N_0.

Hence S_n is bounded.

It remains to show that $\sum_{n=1}^{\infty} z_n = \left(\sum_1^{\infty} x_j\right)\left(\sum_1^{\infty} y_k\right)$

(this is an exercise.)

Definition (Cauchy product)
Given sequences x_j, y_k, define their *Cauchy product* $\{c_l\}_{l=2}^{\infty}$ by

$$c_l = \sum_{j+k=l} x_j y_k = \sum_{j=1}^{l-1} x_j y_{l-j}$$

Corollary. If x_j and y_k are absolutely summable, then

$$\sum_{l=2}^{\infty} c_l = \sum_{j,k=1}^{\infty} x_j y_k = \left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)$$

~~Proof:~~

$c_2 = x_1 y_1$

$c_3 = x_1 y_2 + x_2 y_1$

$c_4 = x_3 y_1 + x_2 y_2 + x_1 y_3$

	x_1	x_2	x_3	x_4
y_1	x_1y_1	x_2y_1	x_3y_1	...
y_2	x_1y_2	x_2y_2	x_3y_2	...
y_3	x_1y_3	x_2y_3	x_3y_3	...
y_4				
$\vdots$				

Proof: Note that c_l is a rearrangement of $\{x_j y_k\}$. By Prev Thm, result follows.