Convergent series $\sum_{n=1}^{\infty} x_n$.

Unconditional convergence
- rearrangement of a sequence.

* Unconditional convergence $\Longleftrightarrow$ absolute convergence.

absolute convergence: $\sum_{n=1}^{\infty} |x_n|$ converges.

Example 5.3. (Geometric series)
Given numbers a and r (not necessarily real), the series $\sum_{k=0}^{\infty} ar^k$ is called a *geometric series* with common ratio r.

Lemma. For any $r \neq 1$,

$$s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

Proof: If $r=1$ then $s_n = n+1$.
Proof is by induction. Leave as an exercise to do it in detail.

$$s_n = 1 + r + r^2 + r^3 + r^4 + \cdots + r^n$$

$$rs_n = \quad r + r^2 + r^3 + r^4 + \cdots + r^n + r^{n+1}$$

$$s_n - rs_n = 1 - r^{n+1} \longrightarrow s_n = \frac{1-r^{n+1}}{1-r}.$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \qquad \frac{d}{dx}(r^{x+1})$$

Theorem. The geometric series $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise (if $a \neq 0$).

Proof: Suppose $|r|<1$. By Lemma,

$$S_n = \sum_{k=0}^{n} ar^k = a\sum_{k=0}^{n} r^k = a\,\frac{1-r^{n+1}}{1-r}.$$

But since $|r|<1$, $r^{n+1} \to 0$ as $n\to\infty$.

Hence

$$\lim_n S_n = \sum_{k=0}^{\infty} ar^k = \lim_n a\cdot\frac{1-r^{n+1}}{1-r} = \frac{a}{1-r}.$$

Suppose $|r|\geq 1$. If $r=1$ then $S_n = a(n+1)$ so clearly S_n does not converge. If $|r|>1$ then we can show that $|S_n|$ is unbounded.

$$|S_n| = |a|\,\frac{|1-r^{n+1}|}{|1-r|} = \frac{|a|}{|1-r|}\,|1-r^{n+1}|$$

$$= \left|\frac{a}{1-r}\right| |r^{n+1}-1| \geq \left|\frac{a}{1-r}\right| (|r|^{n+1}-1)$$

$\boxed{|x-y| \geq \big||x|-|y|\big|}$

But since $|r|>1$, $|r|^{n+1}\to\infty$ as $n\to\infty$. So $|S_n|$ is unbounded.

If $r=-1$ then S_n is bounded but does not converge.

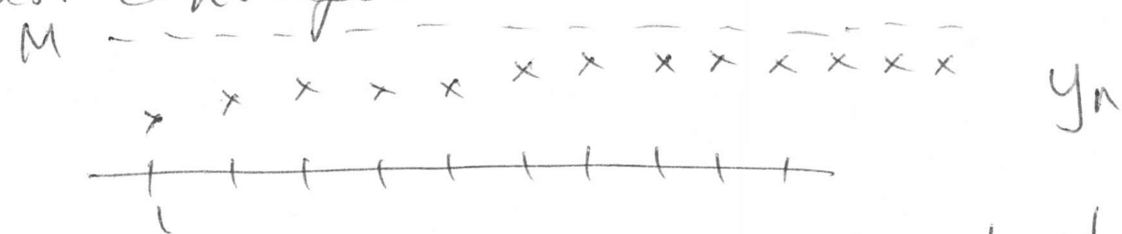
5.2. Convergence Tests.

Theorem. A series $\sum_{n=1}^{\infty} x_n$ of nonnegative terms, that is, for which $x_n \geq 0$ for all n, converges if and only if the sequence of partial sums is bounded.

Bounded: Proof:

$s_n = \sum_{k=1}^{n} x_k$ is bounded means that there is a M such that for all n, $|s_n| \leq M$ (since $x_n \geq 0$ all n, this is same as $s_n \leq M$)

Recall: A bounded increasing sequence in $\mathbb{R}$ must converge.



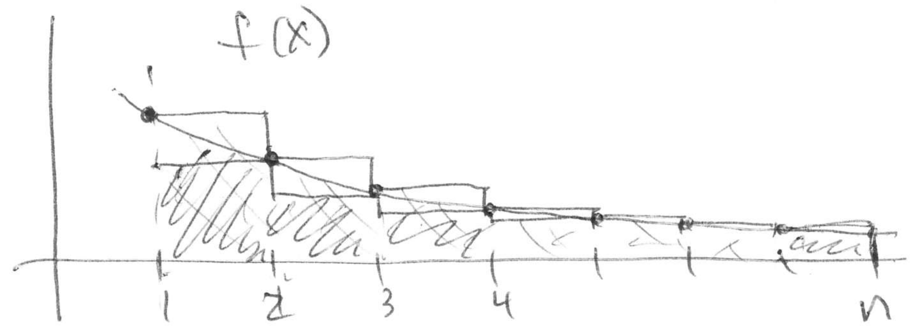
In fact $y_n \to \sup_k \{y_k\}$. = least upper bound.

Recall: Reason for this is that $\mathbb{R}$ is complete, i.e. every bounded set has a supremum.

Theorem 5.2.2. (Integral Test.)
Suppose f is Riemann integrable on $[1,\infty)$, that is, it is integrable on $[1,b]$ for every $b >$ [redacted], is monotone decreasing and approaches 0 as $x \to \infty$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx < \infty$$

Proof: Show that partial sums $\sum_{k=1}^{n} f(k) = S_n$ are bounded iff $\int_1^\infty f(x)dx < \infty$.



$$\sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\,dx \geq \sum_{k=2}^{n} f(k)$$

"only if" direction

"if" direction

"Proof."

Recall: $\limsup_n x_n = \lim_n \left(\sup_{k\ge n} x_k\right)$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \; \{x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \cdots\}$

$n=6$.

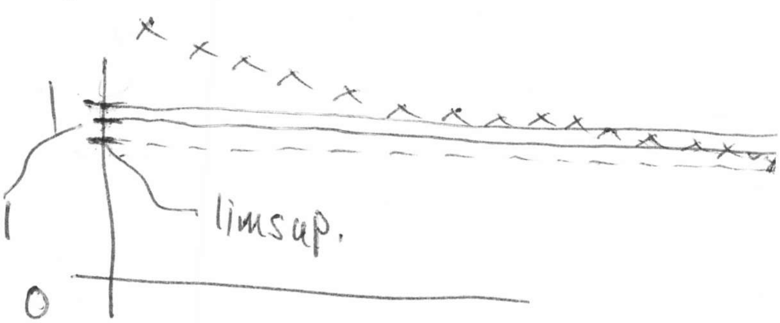
$\left(\sup_{k\ge n} x_k\right) = \sigma_n$ is a decreasing sequence.

so $\lim_n (\sigma_n) = \lim_n \left(\sup_{k\ge n} x_k\right) = \inf_n \sup_{k\ge n} x_k$.

$x_k = (-1)^k \qquad \sup_{k\ge n} x_k = 1$

What if $\limsup_k \frac{x_{k+1}}{x_k} < 1$

$\lim_k \left(\sup_{j\ge k} \frac{x_{j+1}}{x_j}\right) < 1$



This means that for some $r<1$ there is a K such that if $k \ge K$ then $\sup_{j\ge k} \frac{x_{j+1}}{x_j} < r$.

This means $\frac{x_{j+1}}{x_j} < r$ for all $j \ge K$.

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Look at $f(x) = \frac{1}{x^p}$, $\int_1^b \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^b$

$p \neq 1$

$= \frac{b^{1-p}}{1-p} - \frac{1}{1-p}$ etc...

Theorem. (Ratio Test.) Suppose that $x_k > 0$ for all k. If

$$liminf_{k\to\infty} \frac{x_{k+1}}{x_k} > 1$$

then the series $\sum_{n=1}^{\infty} x_n$ diverges. If

$$limsup_{k\to\infty} \frac{x_{k+1}}{x_k} < 1$$

then the series $\sum_{n=1}^{\infty} x_n$ converges.

Basic idea: If $x_k = r^k$ then $\frac{x_{k+1}}{x_k} = r$

So $\liminf \frac{x_{k+1}}{x_k} > 1$ means x_k behaves like r^k with $r > 1$, $\limsup \frac{x_{k+1}}{x_k} < 1$ means x_k behaves like r^k with $r < 1$.

$x_{k+1} < r x_k$, $\quad x_{k+2} < r x_{k+1} < r^2 x_k$,

$x_{k+3} < r x_{k+2} < r^3 x_k$. In general for all j

$j \geq 1$, $\quad x_{k+j} \leq r^j \underline{\underline{x_k}}$.

Therefore,

$$\sum_{k=0}^{N} x_k = \sum_{k=0}^{K-1} x_k + \sum_{k=K}^{N} x_k$$

$$\leq \sum_{k=0}^{K-1} x_k + \sum_{j=0}^{N-K} x_K r^j$$

$$= \sum_{k=0}^{K-1} x_k + x_K \sum_{j=0}^{N-K} r^j$$

$$< \sum_{k=0}^{K-1} x_k + x_K \sum_{j=0}^{\infty} r^j < \infty .$$

Hence partial sums are bounded so $\sum_{k=0}^{\infty} x_k$ converges.

Rem: Why not just say:

$$\limsup_k \frac{x_{k+1}}{x_k} > 1 \Rightarrow \sum_{k=1}^{\infty} x_k \text{ diverges?}$$

e.g. $x_k: 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \dots$

$$\frac{x_{k+1}}{x_k} = 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \dots$$

$$\limsup_k \frac{x_{k+1}}{x_k} = 1$$