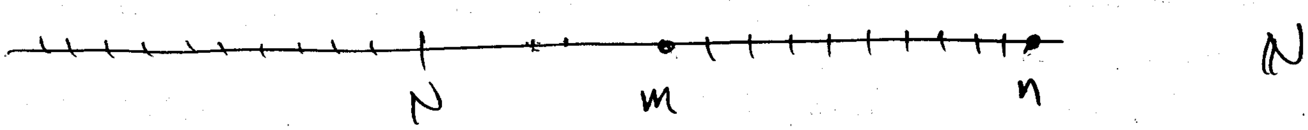
5.7 (c) $\underline{|x_{n+1}-x_n| < r^n|x_1-x_0|}$

Q: If $|x_{n+1}-x_n| \to 0$ is x_k Cauchy? <u>NO</u>

Need: $\underline{|x_n-x_m|<\varepsilon}$ if $n, m \ge N = N(\varepsilon)$.



$$|x_n-x_m| \le |(x_n-x_{n-1})+(x_{n-1}-x_{n-2})+\cdots+(x_{m+1}-x_m)|$$

$$C(r^{n-1}+r^{n-2}+\cdots+r^m)$$

$$\le C \underbrace{\sum_{k=m}^{n-1} r^k}_{\text{convergent series}}$$

Unconditional convergence of $\sum_{k=1}^{\infty} x_k$

$\Updownarrow$

Absolute convergence of $\sum_{k=1}^{\infty} x_k$

Rcl: $\sum x_n$ converges unconditionally if every rearrangement of x_k is summable.

Q: Is every rearrangement summable to same thing? YES

rearrangement: $f : \mathbb{N} \to \mathbb{N}$ bijection

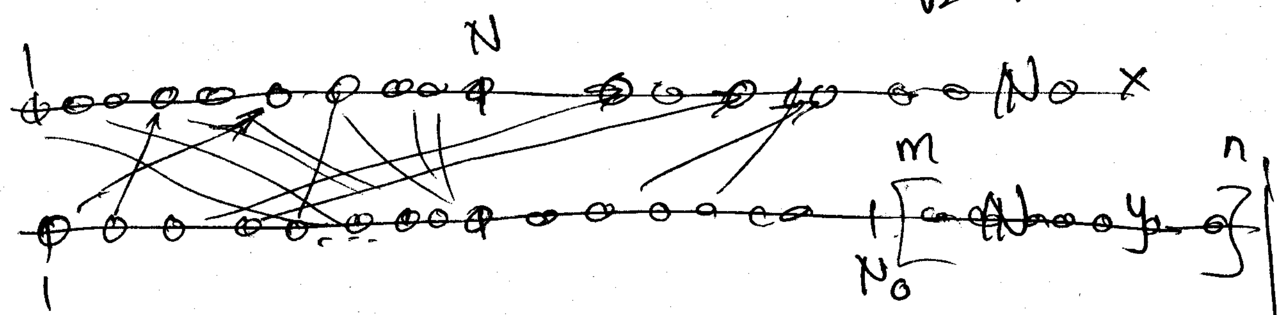
$$x_{f(k)} = y_k.$$

can't do: $y: x_1 \; x_3 \; x_5 \; x_7 \; x_9 \; \cdots$ not a rearrangement.
$\quad\quad x_2 \; x_4 \; x_6 \; x_8 \; x_{10} \; \cdots$

Theorem. A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if and only if it is unconditionally convergent.

Proof. ($\Longrightarrow$) Suppose $\sum_{n=1}^{\infty} x_n$ is absolutely convergent and let y_k be a rearrangement of x_k. Will show $\sum_{k=1}^{\infty} y_k$ converges. Since $\sum_{n=1}^{\infty} x_n$ is abs conv., given $\varepsilon > 0$ there is an N such that if $n, m \geq N$ then $\sum_{k=m}^{n} |x_k| < \varepsilon$.

Show $\exists N_0$ st. $n, m \geq N_0$ then $\left|\sum_{k=m}^{n} y_k\right| < \varepsilon$.



Choose $N_0 \geq N$ so that

$f([1, N_0]) \supseteq [1, N]$, i.e. $\{f(j) : 1 \leq j \leq N_0\} \supseteq \{k : 1 \leq k \leq N\}$

In fact, $N_0 = \max\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(N)\}$.

If $n, m \geq N_0$ then

$$\left|\sum_{k=m}^{n} y_k\right| \leq \sum_{k=m}^{n} |y_k| = \sum_{k \in f([m,n])} |x_k| \leq \sum_{k=N+1}^{N_1} |x_k| < \varepsilon$$

where $N_1 = \max\{f(k) : m \leq k \leq n\}$. Hence $\sum y_k$ converges.

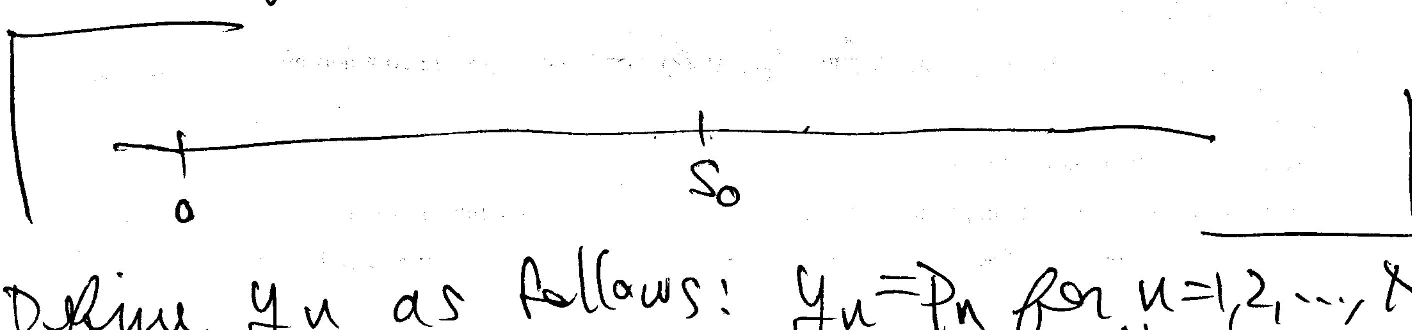
($\Leftarrow$) Suppose $\sum_{n=1}^{\infty} x_n$ is uncond. convergent.
Show $\sum_{n=1}^{\infty} |x_n|$ converges. Assume $\sum_{n=1}^{\infty} |x_n| = \infty$.
Will show $\sum x_n$ is not unconditionally convergent.
We can assume that $\sum_{n=1}^{\infty} x_n$ converges, for if not result is clear. Let

$$x_k^+ = \begin{cases} x_k & \text{if } x_k \geq 0 \\ 0 & \text{if } x_k < 0 \end{cases}, \quad x_k^- = \begin{cases} |x_k| & \text{if } x_k < 0 \\ 0 & \text{if } x_k \geq 0. \end{cases}$$

<u>Claim</u>: The series $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ both diverge.

<u>Pf</u>: Exercise

What we will show is that given $S_0 \in \mathbb{R}$ there is a rearrangement y_n of x_n such that $\sum_{n=1}^{\infty} y_n = S_0$. First divide x_n into two subsequences p_k and m_k, p_k containing the non-negative terms and m_k the negative terms. Define a rearrangement of x_n as follows.

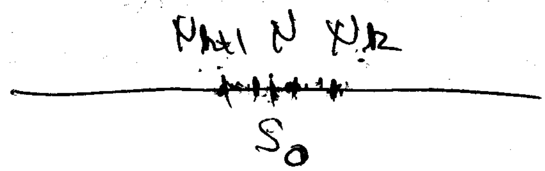


Define y_n as follows: $y_n = p_n$ for $n = 1, 2, \dots, N_1$
where N_1 is first index for which $\sum_{n=1}^{N_1} p_n > S_0$

then define $y_n = m_{n-N_1}$ for $n = N_1+1, \ldots, N_2$ where N_2 is first index for which $\sum_{n=1}^{N_2} y_n < S_0$.

Continue in this fashion and note that at each stage only finitely many terms are required so y_n is a rearrangement of x_n. Also note that $N_k \to \infty$ as $k \to \infty$, and that $N_{k+1} > N_k$. Next note that

$$\left| S_0 - \sum_{n=1}^{N} y_n \right|$$



$$\leq \left| S_0 - \sum_{n=1}^{N_k} y_n \right| \quad \text{where } N_k \leq N < N_{k+1}$$

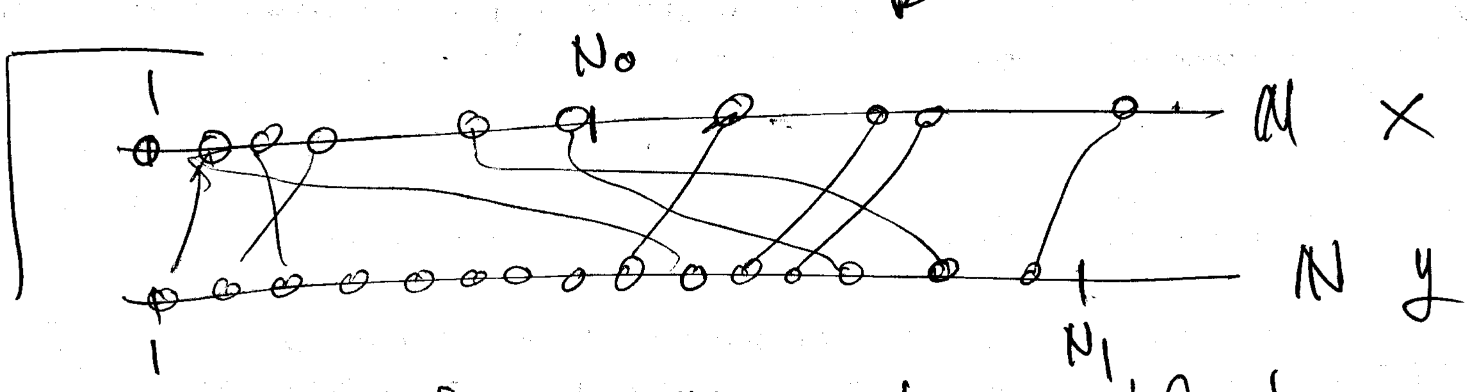
$$\leq \max\left(|m_{N_k}|, |p_{N_k}| \right)$$

Since $N_k \to \infty$, m_{N_k} and $p_{N_k} \to 0$. Therefore

$$\left| S_0 - \sum_{n=1}^{N} y_n \right| \to 0 \text{ as } N \to \infty.$$

Now will prove that if $\sum x_k$ is uncond. convergent then for any rearrangement y_n, $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n$.

Pf: Let $s_n = \sum_{k=1}^{n} x_k$, $t_n = \sum_{k=1}^{n} y_k$. Show that $|s_n - t_n| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ must find N so that if $n \geq N$ then $\left| \sum_{k=1}^{N} (x_k - y_k) \right| < \varepsilon$. Since both x_n and y_n are summable, $\exists N_0$ such that if $n, m \geq N_0$ then $\left| \sum_{k=m}^{n} x_k \right| < \frac{\varepsilon}{2}$ and $\left| \sum_{k=m}^{n} y_k \right| < \frac{\varepsilon}{2}$. But since by prev. theorem, x_k is absolutely summable I can change above to $\sum_{k=m}^{n} |x_k| < \frac{\varepsilon}{2}$.



As before, choose N_1 so large that $\{f(j) : 1 \leq j \leq N_1\} \supseteq \{k : 1 \leq k \leq N_0\}$.

Then if $n \geq N_1$ then

$\sum_{k=1}^{n} (x_k - y_k)$ will contain terms x_k, y_k with $k > N_0$. If $k \leq N_0$ there is a $j \leq N_1$ such that $f(j) = k$ so that $x_k = y_j$ and this term cancels out. So

$$\left| \sum_{k=1}^{n} (x_k - y_k) \right| = \left| \cancel{\sum_{k=1}^{N_0} x_k} + \sum_{k=N_0+1}^{n} x_k - \cancel{\sum_{k \in f^{-1}([1,N_0])} y_k} - \sum_{\substack{k \notin f^{-1}([1,N_0]),\\ 1 \leq k \leq N_1}} y_k \right|$$

$$\leq \left| \sum_{k=N_0+1}^{n} x_k \right| + \left| \sum_{\substack{k \notin f^{-1}([1,N_0])\\ 1 \leq k \leq n}} y_k \right|$$

$$\leq \left| \sum_{k=N_0+1}^{n} x_k \right| + \sum_{k=N_0+1}^{n} |x_k| < \varepsilon.$$