Dirichlet's Test:

$\sum_{n=1}^{\infty} a_n b_n$. If $s_n = \sum_{k=1}^{n} a_k$ is a bounded sequence and $b_n \downarrow 0$ ($b_1 \geq b_2 \geq b_3 \cdots ; b_n \to 0$) then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Idea: $b_n \geq 0$, but $\sum_{n=1}^{\infty} b_n$ need not converge

a_n can contain pos + neg terms that will control partial sums $\sum_{n=1}^{N} a_n b_n$.

Theorem. (Alternating Series Test)
Suppose that $x_n \downarrow 0$. Then the (alternating) series $\sum_{n=1}^{\infty}(-1)^n\, x_n$ converges. Moreover, if $s = \sum_{n=1}^{\infty}(-1)^n\, x_n$ then $|s_n - s| \le x_{n+1}$.

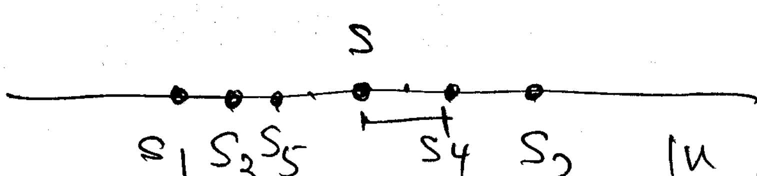
Proof: First part follows from Dirichlet's Test: Let $a_n = (-1)^n$, $b_n = x_n$. We must verify that $p_n = \sum_{k=1}^{n} a_k$ is bounded. But

$$p_n = -1+1-1+1-1+1-1+\cdots\pm 1 = \begin{cases} 0 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases}$$

Hence p_n is bounded ($|p_n| \le 1$ all n) so $\sum_{n=1}^{\infty}(-1)^n x_n$ converges.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges.

$$s_n = \sum_{k=1}^{n}(-1)^k x_k, \quad s = \sum_{k=1}^{\infty}(-1)^k x_k.$$



e.g. $|s - s_4| \le |s_5 - s_4| = x_5$

In general

$$|s - s_n| \le |s_{n+1} - s_n| = x_{n+1}$$

Theorem. (Convergence of trigonometric series - Dirichlet)
Suppose that $a_n \downarrow 0$. Then for every $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges.

Proof: $\sum_{n=1}^{\infty} a_n \sin(nx)$ is a Fourier series or trigonometric series.

e.g. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \rightarrow$ function of x.

eg $x=0$ $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = 0$, $x=\pi$

$\sum_{n=1}^{\infty} \frac{\sin(n\pi)}{n} = 0$, $x = \frac{\pi}{2}$ $\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{2})}{n}$

$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1}$ converges.

For arbitrary x, $\sin(nx)$ is unpredictable

Note that if $x=0$, series converges and that it represents a function with period 2π, so we only need to check $x \in (0, 2\pi)$.
Let $x \in (0, 2\pi)$.
Claim: $S_n = \sum_{k=1}^{n} \sin(kx)$ is bounded.

$e^{it} = \cos(t) + i\sin(t)$

Pf of claim: $\sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$.

Now, $$\sum_{k=0}^{n} e^{ikx} = \sum_{k=0}^{n} (e^{ix})^{k} = \frac{1-e^{i(n+1)x}}{1-e^{ix}}$$

Therefore,

$$\sum_{k=0}^{n} \sin(kx) = \frac{1}{2i}\left[\sum_{k=0}^{n} e^{ikx} - \sum_{k=0}^{n} e^{-ikx}\right]$$

$$= \frac{1}{2i}\left[\frac{1-e^{i(n+1)x}}{1-e^{ix}} - \frac{1-e^{-i(n+1)x}}{1-e^{-ix}}\right]$$

Note that $|1-e^{ix}| = |(e^{-ix/2} - e^{ix/2})(e^{ix/2})|$

$= |e^{-ix/2} - e^{ix/2}| = 2\left|\sin\left(\frac{x}{2}\right)\right|$

Also $|1-e^{-ix}| = 2\left|\sin\left(\frac{x}{2}\right)\right|$

Finally,

$$\left|\sum_{k=0}^{n} \sin(kx)\right| \le \left|\frac{1}{2i}\right| \left[\frac{\overbrace{|1-e^{i(n+1)x}|}^{\le 2}}{2\left|\sin\left(\frac{x}{2}\right)\right|} + \frac{\overbrace{|1-e^{-i(n+1)x}|}^{\le 2}}{2\left|\sin\left(\frac{x}{2}\right)\right|}\right]$$

$$\le \left(\frac{1}{2}\right)\frac{4}{2\left|\sin\left(\frac{x}{2}\right)\right|} = \frac{1}{\left|\sin\left(\frac{x}{2}\right)\right|} < \infty \text{ for each } x \in (0, 2\pi).$$

Therefore result follows from Dirichlet's test.

5.1 Series of Constants (continued).

Definition 5.1.2 (Absolute convergence) Let x_n be a sequence of numbers. The series $\sum_{n=1}^{\infty} x_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} |x_n|$ converges. In this case, we say that the sequence x_n is *absolutely summable.* A series that is convergent but not absolutely convergent is called *conditionally convergent.*

Theorem 5.1.3. Every absolutely summable sequence is summable.

Proof: Look at Cauchy criterion: Given $\varepsilon > 0$, find N so that if $n, m \geq N$ then $\left|\sum_{k=m}^{n} x_k\right| < \varepsilon$. Note that $\left|\sum_{k=m}^{n} x_k\right| \leq \sum_{k=m}^{n} |x_k|$. Since $|x_n|$ is summable there is an N so that if $m, n \geq N$ then $\sum_{k=m}^{n} |x_k| < \varepsilon$. This same N works for x_n.

Rearrangement

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad \cdots \quad x_{62}$

$y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5$

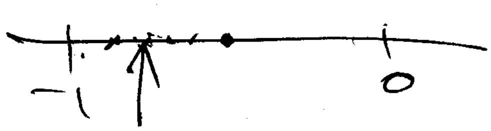
$f(1) = 4, \; f(2) = 6, \; f(3) = 62, \; \cdots$

How could a rearrangement fail to be summable?

e.g. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is conditionally convergent.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$



$= \frac{1}{2} + \frac{1}{4} + \quad + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{1000000}$

-1

$+ \frac{1}{1000002} + \cdots + \frac{1}{\quad}$

Definition. (Unconditional convergence.)
A sequence y_n is a *rearrangement* of a sequence x_n if there is a bijection $f: \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{f(n)}$. A series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if every rearrangement y_n of x_n is summable.

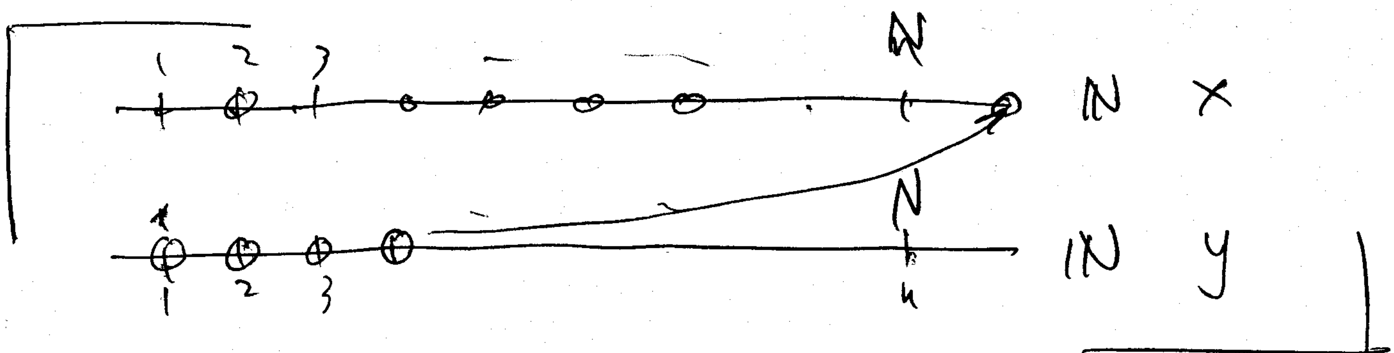
Lemma. If $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* then for every rearrangement y_n,

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n$$

Proof:

Let $s_n = \sum_{k=1}^{n} x_k$, $t_n = \sum_{k=1}^{n} y_k$. We will show that $\lim_{n\to\infty} |s_n - t_n| = 0$. This will be sufficient.

Let $\varepsilon > 0$.



~~Let~~ Since x_n, y_n are summable there is an N such that if $n, m \geq N$ then

$$\left| \sum_{k=m}^{n} x_k \right| < \frac{\varepsilon}{2} \text{ and } \left| \sum_{k=m}^{n} y_k \right| < \frac{\varepsilon}{2}.$$

Finish next time