5.1 Series of Constants.

Definition 5.1.1 (Convergent series) Let x_n be a sequence of numbers. The *(infinite) series* $\sum_{n=1}^{\infty} x_n$ *converges to* s if the sequence of partial sums, $s_n = \sum_{k=1}^{n} x_k$ converges to s. In this case, we say that the sequence of terms x_n is *summable* and that the corresponding series is *convergent.* Otherwise, we say that the series is *divergent.*

Lemma. (Cauchy criterion.) The series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums, $s_n = \sum_{k=1}^{n} x_k$ is Cauchy, that is, given $\epsilon > 0$, there is an N such that if $n, m \geq N$ then $|s_n - s_{m-1}| = |\sum_{k=m}^{n} x_k| < \epsilon$.

Theorem 5.1.1 (n^{th} term test)
If x_n is a summable sequence, then $x_n \to 0$.

Proof. This follows immediately from the Cauchy criterion. Why? Let $\varepsilon>0$, choose N in the Cauchy criterion. Then take $n=m \geq N$. Then $|S_n - S_{m-1}| = |S_n - S_{n-1}| = |x_n| < \varepsilon$ whenever $n \geq N$. Hence $x_n \to 0$.

A sequence x_n is <u>Cauchy</u> if $\forall \varepsilon > 0$ there is an N such that $n, m \geq N$ implies $|x_n - x_m| < \varepsilon$.

Fact: x_n converges iff it is Cauchy.

<u>For series:</u> $S_n = \sum_{k=1}^{n} x_k$

$$|S_n - S_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m+1}^{n} x_k \right|$$

if $n > m$.

Will typically write:

$$\left| \sum_{k=m}^{n} x_k \right| \to 0 \text{ as } n, m \to \infty$$

as shorthand for "S_n is Cauchy".

<u>e.g.</u> $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ $\quad S_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$ Guess: $S_n \to 2$

Look at $|S_n - 2|$. Know $S_n = \dfrac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}$

$= 2 - \left(\frac{1}{2}\right)^n$. $\quad |S_n - 2| = \left|2 - \left(\frac{1}{2}\right)^n - 2\right| = \left(\frac{1}{2}\right)^n \to 0$

as $n \to \infty$.

Theorem: The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof: (Dates back to 1200s)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$= 1 + \overset{k=0}{\frac{1}{2}} + \overset{k=1}{\left(\frac{1}{3} + \frac{1}{4}\right)} + \overset{k=2}{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)} +$$

$$\overset{k=3}{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)} + \cdots + \left(\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right)$$

$$+ \left(\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

The partial sums diverge to ∞ but <u>very slowly</u>.

We have all seen: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$

<u>Theorem.</u> (Abel's Formula or Summation by Parts.)
Let a_n and b_n be real-valued sequences and let $A_n = \sum_{k=1}^{n} a_k$. Then for all $n > 1$

$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)$$

<u>Proof:</u> Int by parts: $\int_a^b f(x) g(x) dx = F(x) g(x) \Big|_a^b - \int_a^b F(x) g'(x) dx$

Lets mimic the proof of int by parts.

$$\int_a^b (fg)' = \int_a^b f'g + \int g'f = f(b)g(b) - f(a)g(a)$$

Consider sequences x_n, y_n. (say $x_0 = y_0 = 0$)

$$x_n y_n - x_{n-1} y_{n-1} = x_n y_n - x_n y_{n-1} + x_n y_{n-1} - x_{n-1} y_{n-1}$$

$$= x_n (y_n - y_{n-1}) + y_{n-1} (x_n - x_{n-1})$$

$$x_n y_n = \sum_{k=1}^{n} (x_k y_k - x_{k-1} y_{k-1})$$

$$= \sum_{k=1}^{n} x_k (y_k - y_{k-1}) + \sum_{k=1}^{n} y_{k-1} (x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} x_k (y_k - y_{k-1}) + \sum_{k=1}^{n-1} y_k (x_{k+1} - x_k)$$

Let $y_n = A_n = \sum_{k=1}^{n} a_k$, $x_n = b_n$.

Note that $y_k - y_{k-1} = a_k$. Hence we have

$$A_n b_n = \sum_{k=1}^{n} b_k a_k + \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)$$

as required.

Theorem. (Dirichlet's Test)
Let a_n and b_n be real-valued sequences and suppose that $s_n = \sum_{k=1}^{n} a_k$ is a bounded sequence and that $b_n \downarrow 0$ (that is, the sequence b_n is decreasing and converges to 0). Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof: Since S_n is bounded, let's say $|S_n| \le B$ for all n. By Abel's formula,

$$\sum_{n=1}^{N} a_n b_n = S_N b_N - \sum_{n=1}^{N-1} s_n (b_{n+1} - b_n)$$

Therefore,

$$\left|\sum_{n=M}^{N} a_n b_n\right| = \left| S_N b_N - S_{M-1} b_{M-1} - \sum_{n=M}^{N-1} s_n (b_{n+1} - b_n) \right|$$

$$\le |S_N b_N - S_{M-1} b_{M-1}| + \left|\sum_{n=M}^{N-1} s_n (b_{n+1} - b_n)\right|$$

$$\le |S_N b_N| + |S_{M-1} b_{M-1}| + \left|\sum_{n=M}^{N-1} s_n (b_{n+1} - b_n)\right|$$

$$\le B(b_N + b_{M-1}) + \sum_{n=M}^{N-1} |S_n| (b_{n+1} - b_n)$$

$$\le B(b_N + b_{M-1}) + B \sum_{n=M}^{N-1} (b_{n+1} - b_n)$$

$$\le B(b_N + b_{M-1}) + B(b_N - b_M)$$

Since $b_n \to 0$ as $n \to \infty$, given $\varepsilon > 0$, there is an N_0 such that if $n \geq N_0$ then $|b_n| < \frac{\varepsilon}{4B}$. Hence if $N, M \geq N_0 + 1$ then

$$\left|\sum_{n=M}^{N} a_n b_n\right| \le B(b_N + b_{M-1}) + B(b_N - b_M)$$

$$\le B(b_N + b_{M-1} + b_N + b_M)$$

$$< B\left(\frac{4\varepsilon}{4B}\right) = \varepsilon.$$

Or I could just say

$$\le B(b_N + b_{M-1}) + B(b_N - b_M) \longrightarrow 0$$

as $N, M \to \infty$

as shorthand for above.