ules for Differentiation

Examples

1) The product rule: $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$

How to prove?

Directly:

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} g(a+h)\left[\frac{f(a+h)-f(a)}{h}\right] + f(a)\left[\frac{g(a+h)-g(a)}{h}\right]$$

$$= g(a)f'(a) + f(a)g'(a).$$

2) The Chain Rule.

Spse f is differentiable at $a \in \mathbb{R}$ and g at $f(a)$. Then $g\circ f$ is differentiable at a with $(g\circ f)'(a) = g'(f(a))\,f'(a)$.

How to prove?

Directly:

$$\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{h}$$

$$= \lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)} \cdot \frac{f(a+h)-f(a)}{h}$$

$= g'(f(a))f'(a)$ since $f(a+h)-f(a) \to 0$ as $h \to 0$

Problem: Not valid whenever $f(a+h)-f(a)=0$. which could happen for lots of h. Can be gotten around but there is a better way.

Idea: Let $F(x) = g\circ f(x) = g(f(x))$

Need to show there is a $T_F(x) = mx$ satisfying

$$\lim_{h\to 0} \frac{|F(a+h)-F(a)-T_F(h)|}{|h|} = 0.$$

Our claim is that $T_F(x) = g'(f(a))f'(a)x$. So lets plug it in: Need to show

$$\lim_{h\to 0} \frac{|g(f(a+h))-g(f(a))-g'(f(a))f'(a)h|}{|h|} = 0$$

We know:

$$\lim_{s\to 0} \frac{|g(f(a)+s)-g(f(a))-g'(f(a))s|}{|s|} = 0$$

and

$$\lim_{r \to 0} \frac{|f(a+r) - f(a) - f'(a) r|}{|r|} = 0.$$

So write. LHS =

$$|g(f(a+h)) - g(f(a)) - g'(f(a))(f(a+h) - f(a))$$
$$+ g'(f(a))(f(a+h) - f(a)) - g'(f(a)) f'(a) h|$$

$$\leq |g(f(a+h)) - g(f(a)) - g'(f(a))(f(a+h) - f(a))|$$
$$+ |g'(f(a))| \, |f(a+h) - f(a) - f'(a) h| = \mathrm{I} + \mathrm{II}$$

Need ~~to~~ show that $\frac{(\text{above})}{|h|} \to 0$ ~~as~~ $h \to 0$.

Looking II

$$\lim_{h \to 0} |g'(f(a))| \frac{|f(a+h) - f(a) - f'(a) h|}{h} = 0$$

since $|g'(f(a))|$ is constant.

Looking at I let $\varepsilon > 0$, There is a $\delta_0 > 0$ such that if $|f(a+h) - f(a)| < \delta_0$ then $|g(f(a+h)) - g(f(a)) - g'(f(a))(f(a+h) - f(a))| < \varepsilon |f(a+h) - f(a)|$.

Since f is continuous at a there is a $\delta_1 > 0$ such that if $|h| < \delta_1$ then $|f(a+h) - f(a)| < \delta_0$

Hence for all $|h| < \delta_1$,

$$\frac{|g(f(a+h)) - g(f(a)) - g'(f(a))(f(a+h) - f(a))|}{|h|}$$

$$< \varepsilon \frac{|f(a+h) - f(a)|}{|h|}$$

Taking lim sup of both sides as $h \to 0$,

$$\limsup_{h \to 0} \text{LHS} \leq \varepsilon \limsup_{h \to 0} \frac{|f(a+h) - f(a)|}{|h|}$$

$$= \varepsilon \lim_{h \to 0} \frac{|f(a+h) - f(a)|}{|h|} = \varepsilon f'(a).$$

Since $\varepsilon > 0$ was arbitrary, $\limsup \text{LHS} \leq 0$

Since clearly $0 \leq \liminf \text{LHS}$ we have

$\liminf \text{LHS} = \limsup \text{LHS} = 0$.

Hence $\lim \text{LHS} = 0$, as required.