MATH 315 – HOMEWORK 10
SOLUTIONS TO SELECTED EXERCISES

Section 5.3, Exercise 8.

Define the function $F(x) = \alpha \int_a^x f(t)\,dt + \beta \int_x^b f(t)\,dt$. Then by FTC I, F is continuously differentiable on $[a, b]$ and by hypothesis, $F(x) = 0$ for all $x \in [a, b]$. In particular, F is constant so again by FTC I, $F'(x) = \alpha\, f(x) - \beta\, f(x) = (\alpha - \beta)\, f(x) = 0$ for all $x \in [a, b]$. Since $\alpha \neq \beta$ it follows that $f(x) = 0$ for all $x \in [a, b]$.

Exercise 10.

(a). ($\Longleftarrow$) Suppose that f is constant on $[a, b]$ and that say $f(x) = M$ for all $x \in [a, b]$. Then by FTC II, $\int_a^b f(x)g'(x)\,dx = M \int_a^b g'(x)\,dx = M(g(b) - g(a)) = 0$ since $g(b) = g(a) = 0$.

($\Longrightarrow$) By the Integration by Parts formula, $\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx = 0$ by the hypotheses of the theorem. Since f is increasing, $f'(x) \geq 0$ on $[a, b]$. By the Second Mean Value Theorem there is an $x_0 \in [a, b]$ such that

$$0 = \int_a^b f'(x)g(x)\,dx = (\sup_{x\in[a,b]} f'(x)) \int_{x_0}^b g(x)\,dx.$$

Since $g(x) > 0$ on $[a, b]$, the Comparison Theorem implies that $\int_{x_0}^b g(x)\,dx > 0$, so that $\sup_{x\in[a,b]} f'(x) = 0$. Since $f'(x) \geq 0$ on $[a, b]$ this implies that $f'(x) = 0$ on $[a, b]$ and since f is continuously differentiable, f is constant.

Section 5.4, Exercise 6.

(a). First note that since $f(x) \geq 0$, the integral $\int_a^d f(x)\,dx$ is an increasing function of d and therefore if we can show that it is bounded for $d \in [a, b)$ it will follow that the limit exists as a finite number.

The hypothesis implies that there is a number $a < d < b$ such that $0 \leq f(x)/g(x) < L + 1$ whenever $d < x < b$ (this follows from the definition of convergence when $\epsilon = 1$). Therefore for all such x, $0 \leq f(x) \leq (L + 1)\, g(x)$ and the Comparison Theorem implies that for all $a < d < d' < b$

$$0 \leq \int_a^{d'} f(x)\,dx = \int_a^d f(x)\,dx + \int_d^{d'} f(x)\,dx \leq \int_a^d f(x)\,dx + (L+1) \int_d^{d'} g(x)\,dx \leq \int_a^d f(x)\,dx + (L+1)$$

since $g(x) \geq 0$ on $[a, b)$ and hence $\int_d^{d'} g(x)\,dx$ is increasing as a function of d'. Since each term in the sum on the right is a finite number, $\int_a^d f(x)\,dx$ is bounded as a function of d and hence $\lim_{d\to b^-} \int_a^d f(x)\,dx$ exists and is finite.

(b). The proof of part (b) is similar to part (a). Since $L > 0$ there is a number $a < d < b$ such that if $d < x < b$ then $f(x)/g(x) > L/2$ (this follows from the definition of convergence when $\epsilon = L/2$). Therefore for all such x, $f(x) > (L/2)g(x)$. We now proceed similarly to the proof above making use of this inequality to show that since $\lim_{d\to b^-}\int_a^d g(x)\,dx = \infty$ then $\lim_{d\to b^-}\int_a^d f(x)\,dx = \infty$ and f is not improperly integrable on $[a, b]$.