MATH 315 – HOMEWORK 9
SOLUTIONS TO SELECTED EXERCISES

Section 5.1, Exercise 2.

(b). Suppose that (*) holds. We will show that for each n, $L(f, P_n) \leq \int_0^1 f(x)\,dx \leq U(f, P_n)$. To see this, note that $\int_0^1 f(x)\,dx = \inf_P U(f, P) \leq U(f, P_n)$ since P_n is a partition of $[0, 1]$ and that $L(f, P_n) \leq \sup_P L(f, P) = \int_0^1 f(x)\,dx$ for the same reason. Therefore,

$$L(f, P_n) - I_0 \leq \int_0^1 f(x)\,dx - I_0 \leq U(f, P_n) - I_0$$

and since $\lim_{n\to\infty} L(f, P_n) - I_0 = \lim_{n\to\infty} U(f, P_n) - I_0 = 0$, the Squeeze Theorem implies that $\int_0^1 f(x)\,dx - I_0 = 0$ as desired.

Exercise 5.

($\Longleftarrow$) Clearly if $f(x) = 0$ for all $x \in [a, b]$ then $\int_a^c f(x)\,dx = 0$ for all $c \in [a, b]$.

($\Longrightarrow$) Suppose that for some $x_0 \in [a, b]$, $f(x_0) \neq 0$, and let us assume for simplicity that $f(x_0) > 0$. Since f is continuous at x_0 and by previous results, there is an interval $[c, d] \subseteq [a, b]$ containing x_0 such that $f(x) > 0$ on $[c, d]$. From this it follows that $\int_c^d f(x)\,dx > 0$ since if $P = \{c = x_0, x_1, \ldots, x_n = d\}$ is a partition of $[c, d]$ then for each j, $m_j(f) = \inf_{x\in[x_{j-1},x_j]} f(x) > 0$ and so $L(f, P) = \sum_{j=1}^n m_j(f)(x_j - x_{j-1}) > 0$ which implies that $\int_c^d f = \sup_P L(f, P) > 0$. Finally we have that

$$\int_c^d f(x)\,dx = \int_a^d f(x)\,dx - \int_a^c f(x)\,dx > 0$$

which implies that at least one of $\int_a^d f$ or $\int_a^c f$ is not zero, contradicting the hypothesis of the theorem. This proves the result by proving its contrapositive.

(Note that if we were able to use the FTC then this result becomes trivial since by defining $F(x) = \int_a^x f(x)\,dx$ then by hypothesis, $F(x) = 0$ for all $x \in [a, b]$. By FTC, $F'(x) = f(x) = 0$ for all $x \in [a, b]$.)

Exercise 8.

(a). Since f is increasing on $[a, b]$ it follows that for any subinterval $[c, d]$ of $[a, b]$ that $\sup_{x\in[c,d]} f(x) = f(d)$ and $\inf_{x\in[c,d]} f(x) = f(c)$. Given a partition

$P = \{a = x_0, x_1, \ldots, x_n = b\}$, it follows that for each j, $M_j(f) = f(x_j)$ and $m_j(f) = f(x_{j-1})$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n}(M_j(f) - m_j(f))(x_j - x_{j-1}) &= \sum_{j=1}^{n}(f(x_j) - f(x_{j-1}))(x_j - x_{j-1}) \\ &\leq \sup_{1\leq j\leq n}(x_j - x_{j-1})\sum_{j=1}^{n} f(x_j) - f(x_{j-1}) \\ &= \|P\|(f(b) - f(a))\end{aligned}$$

since the last sum telescopes to this value.

(b). Given $\epsilon > 0$ choose a partition P of $[a, b]$ such that $\|P\| < \epsilon/(f(b) - f(a))$ (if f is not identically zero then $f(b) - f(a) > 0$ and if f is identically zero it is clearly integrable). Then by the inequality above,

$$U(f,P)-L(f,P) = \sum_{j=1}^{n}(M_j(f)-m_j(f))(x_j-x_{j-1}) < \epsilon/(f(b)-f(a))\,(f(b)-f(a)) = \epsilon.$$

Section 5.2, Exercise 2.

We will give two proofs for $f \vee g$ and in each case a similar proof works for $f \wedge g$. The first proof is a direct one (and hence more complicated) and the second uses Exercise 9, p. 65.

(1) We will prove the result for $f \vee g = \max(f, g)$ by showing that for any partition $P\{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$ and any index j,

$$\begin{aligned}M_j(f \vee g) - m_j(f \vee g) &\leq \max(M_j(f) - m_j(f), M_j(g) - m_j(g)) \\ &\leq (M_j(f) - m_j(f)) + (M_j(g) - m_j(g)).\end{aligned}$$

To see this fix j and let $I = [x_{j-1}, x_j]$.

We will show first that $M_j(f \vee g) = \max(M_j(f), M_j(g))$. Since $f \vee g(x) \geq f(x)$ and $f \vee g(x) \geq g(x)$ for all $x \in I$ it follows that $M_j(f \vee g) \geq M_j(f)$ and $M_j(f \vee g) \geq M_j(g)$ and hence $M_j(f \vee g) \geq \max(M_j(f), M_j(g))$. If $M_j(f \vee g) > \max(M_j(f), M_j(g))$ then by the approximation property for suprema, there would be an $x_0 \in I$ such that $f \vee g(x_0) > f(x)$ and $f \vee g(x_0) > g(x)$ for all $x \in I$. But since $f \vee g(x_0) = f(x_0)$ or $g(x_0)$, this is impossible.

Since $f \vee g(x) \geq f(x)$ and $f \vee g(x) \geq g(x)$ for all $x \in I$ it follows that $m_j(f \vee g) \geq m_j(f)$ and $m_j(f \vee g) \geq m_j(g)$ and hence that $m_j(f \vee g) \geq \max(m_j(f), m_j(g))$. If $M_j(f \vee g) = M_j(f)$ then $M_j(f \vee g) - m_j(f \vee g) \leq M_j(f) - m_j(f) \leq \max(M_j(f) - m_j(f), M_j(g) - m_j(g))$ and if $M_j(f \vee g) = M_j(g)$ then $M_j(f \vee g) - m_j(f \vee g) \leq M_j(g) - m_j(g) \leq \max(M_j(f) - m_j(f), M_j(g) - m_j(g))$.

Note finally that $\max(M_j(f) - m_j(f), M_j(g) - m_j(g)) \leq (M_j(f) - m_j(f)) + (M_j(g) - m_j(g))$ since both terms are nonnegative.

Now given $\epsilon > 0$ there are partitions P and Q of $[a, b]$ with the property that $U(f, P) - L(f, P) < \epsilon/2$ and $U(g, Q) - L(g, Q) < \epsilon/2$. Taking $R = P \cup Q$ it follows that $U(f, R) - L(f, R) < \epsilon/2$ and $U(g, R) - L(g, R) < \epsilon/2$ (we have made this argument in detail in class; check your notes!). Therefore

$$\begin{aligned}
& U(f \vee g, R) - \text{Ł}(f \vee g, R) \\
= & \sum_{j=1}^{n} (M_j(f \vee g) - m_j(f \vee g))(x_{j-1} - x_j) \\
\leq & \sum_{j=1}^{n} [(M_j(f) - m_j(f)) + (M_j(g) - m_j(g))](x_{j-1} - x_j) \\
= & \sum_{j=1}^{n} (M_j(f) - m_j(f))(x_{j-1} - x_j) + \sum_{j=1}^{n} (M_j(g) - m_j(g))(x_{j-1} - x_j) \\
< & \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

(2). By Exercise 9, p. 65, $f \vee g = [(f + g) + |f - g|]/2$. Since we have already established in class and by results in the text that if f and g are integrable then so is $f + g$ and $f - g$ and since the latter is integrable so is $|f - g|$, it follows that $(1/2)[(f + g) + |f - g|]$ is also integrable.

Exercise 5.

(a). Since f is integrable on $[a, b]$ it is bounded there so there is an $M \in \mathbf{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Therefore for all $a \in [a, b]$ and all $n \in \mathbf{N}$, $0 \leq |f(x)\, g_n(x)| \leq M\, |g_n(x)| = M\, g_n(x)$ since $g_n(x) \geq 0$. By the Comparison Theorem,

$$\left| \int_a^b f(x)\, g_n(x)\, dx \right| \leq \int_a^b |f(x)\, g_n(x)|\, dx \leq M \int_a^b |g_n(x)|\, dx = M \int_a^b g_n(x)\, dx.$$

Therefore, by the Squeeze Theorem, $\lim_{n\to\infty} \int_a^b f(x)\, g_n(x)\, dx = 0$.

(b). Note that for all $x \in [0, 1]$ and $n \in \mathbf{N}$, $x^n \geq 0$ and that $\int_0^1 x^n\, dx = 1/(n+1) \to 0$ as $n \to \infty$. Hence the result follows from part (a) using $g_n(x) = x^n$.