MATH 315 – HOMEWORK 7
SOLUTIONS TO SELECTED EXERCISES

Section 4.1, Exercise 3.

(a). That $f(c) \geq f(x)$ whenever $|x - c| < \delta$ is equivalent to the statement that $f(c + h) - f(c) \leq 0$ whenever $0 < h < \delta$ and $f(c + H) - f(c) \leq 0$ whenever $-\delta < H < 0$. This implies that $(f(c+h) - f(c))/h \leq 0$ whenever $0 < h < \delta$ and $(f(c+H) - f(c))/H \geq 0$ whenever $-\delta < H < 0$.

(b). Since f is differentiable at c, $f'(c) = \lim_{h\to 0^+}(f(c + h) - f(c))/h \leq 0$ and $f'(c) = \lim_{H\to 0^-}(f(c + H) - f(c))/H \geq 0$ by part (a). Therefore, $f'(c) = 0$.

(c). f has a *local minimum* at c if and only if there is a $\delta > 0$ such that $f(x) \geq f(c)$ for all $|x - c| < \delta$. In this case, $(f(c+h) - f(c))/h \geq 0$ whenever $0 < h < \delta$ and $(f(c+H) - f(c))/H \leq 0$ whenever $-\delta < H < 0$. If f is differentiable at c then $f'(c) = 0$. The proof of these results is almost identical to the proofs in parts (a) and (b).

(d). Let $f(x) = x^3$ on $\mathbf{R}$. Then $f'(x) = 3x^2$ for all x so that $f'(0) = 0$ but clearly 0 is neither a local maximum nor local minimum for f.

Exercise 7.

Since $|x|^\alpha \sin(1/x)$ is continuous for $\alpha > 0$ and $x \neq 0$ (this is true because the function is a product of composites of continuous functions), we need only check that $\lim_{x\to 0} f_\alpha(x) = f_\alpha(0) = 0$. But if $x \neq 0$, $|f_\alpha(x)| = |x|^\alpha\, |sin(1/x)| \leq |x|^\alpha$ and since $\alpha > 0$, $\lim_{x\to 0} |x|^\alpha = 0$. Hence by the squeeze theorem, $\lim_{x\to 0} f_\alpha(x) = 0$.

Let $\alpha > 1$. Then if $x > 0$, $f_\alpha(x) = x^\alpha \sin(1/x)$ and by the usual rules of differentiation $f'_\alpha(x) = -x^{\alpha-2} \cos(1/x) + \alpha x^{\alpha-1} \sin(1/x)$, and if $x < 0$, $f_\alpha(x) = (-x)^\alpha \sin(1/x)$ and $f'_\alpha(x) = (-x)^{\alpha-2} \cos(1/x) - \alpha(-x)^{\alpha-1} \sin(1/x)$. Hence f_α is differentiable when $x \neq 0$. If $x = 0$, then

$$f'_\alpha(0) = \lim_{h\to 0} \frac{f_\alpha(h) - f_\alpha(0)}{h} = \lim_{h\to 0} \frac{|h|^\alpha \sin(1/h)}{h} == \lim_{h\to 0}(|h|/h)\, |h|^{\alpha-1} \sin(1/h) = 0$$

since $|h|/h \sin(1/h)$ is bounded for $h \neq 0$ and $\lim_{h\to 0} |h|^{\alpha-1} = 0$ since $\alpha > 1$. Therefore f_α is differentiable on $\mathbf{R}$.

Section 4.2, Exercise 5.

Since f is differentiable at a,

$$\begin{aligned}
\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\,\frac{f(a) - f(a+h)}{f(a)\,f(a+h)} \\
&= -\left(\lim_{h\to 0} \frac{1}{f(a)\,f(a+h)}\right)\left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right) \\
&= \frac{1}{f^2(a)}\,f'(a).
\end{aligned}$$

Exercise 7.

The proof will be by induction on n. If $n = 1$ then

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a) = \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(a)g^{(n-k)}(a)$$

since $\binom{1}{0} = \binom{1}{1} = 1.$

Now assume that the formula holds for n.

$$\begin{aligned}
(fg)^{(n+1)}(a) &= \frac{d}{dx}(fg)^{(n)}(a) \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx} f^{(k)}(\cdot)g^{(n-k)}(\cdot)\Big|_{x=a} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)}(a)g^{(n-k)}(a) + f^{(k)}(a)g^{(n-k+1)}(a)\right) \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(a)g^{(n-k+1)}(a) \\
&\qquad + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a) \\
&= \binom{n}{0} f(a)g^{(n+1)}(a) + \binom{n}{n} f^{(n+1)}(a)g(a) \\
&\qquad + \sum_{k=1}^{n+1}\left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)}(a)g^{(n-k+1)}(a) \\
&= \binom{n+1}{0} f(a)g^{(n+1)}(a) + \binom{n+1}{n+1} f^{(n+1)}(a)g(a)
\end{aligned}$$

$$+\sum_{k=1}^{n+1}\begin{pmatrix} n+1 \\ k \end{pmatrix} f^{(k)}(a)g^{(n-k+1)}(a)$$

$$= \sum_{k=0}^{n+1}\begin{pmatrix} n+1 \\ k \end{pmatrix} f^{(k)}(a)g^{(n-k+1)}(a)$$

where in the fourth equality we made the change of index $k \mapsto k-1$ and in the sixth equality have used the fact, proved in Chapter 1, that $\begin{pmatrix} n \\ k-1 \end{pmatrix} + \begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n+1 \\ k \end{pmatrix}$, and the fact that $\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} n+1 \\ 0 \end{pmatrix} = 1$ and that $\begin{pmatrix} n \\ n \end{pmatrix} = \begin{pmatrix} n+1 \\ n+1 \end{pmatrix} = 1$.

Section 4.3, Exercise 3.

(a). If $x \neq 0$ then by the chain rule, $f'(x) = d/dx(e^{-1/x^2}) = 2x^{-3}\, e^{-1/x^2}$ which is a continuous function for $x \neq 0$. The only issue then is to show that $f'(0)$ exists and that $\lim_{x\to 0} f'(x) = f'(0)$. Now, $f'(0) = \lim_{h\to 0}(f(h) - f(0))/h = \lim_{h\to 0} e^{-1/h^2}/h$. Since $-1/h^2 \to -\infty$ as $h \to 0$, $e^{-1/h^2} \to 0$ as $h \to 0$. Since $e^x > x+1$ for $x > 0$, $e^{1/h^2} > (1/h^2) + 1$ and $e^{-1/h^2} < ((1/h^2)+1)^{-1} = h^2/(1+h^2)$ for $h \in \mathbf{R}$. Therefore, $|e^{-1/h^2}/h| < |h|/(1+h^2)$ and it follows that $\lim_{h\to 0} e^{-1/h^2}/h = 0$. Hence $f'(0) = 0$.

To see that $\lim_{x\to 0} f'(x) = \lim_{h\to 0} 2h^{-3}\, e^{-1/h^2} = 0$, we will examine the larger question of evaluating $\lim_{h\to 0} h^{-n}\, e^{-1/h^2}$, for $n \in \mathbf{N}$. By making the change of variables $x = h^{-1}$ we see that the limit in question becomes $\lim_{x\to\pm\infty} x^n e^{-x^2}$. Repeated application of L'Hopital's rule shows that the value of this limit is always 0. Hence f' is continuous at 0 and hence on all of $\mathbf{R}$.

(b). The question of higher derivatives for f will be dealt with somewhat informally. First observe that by the usual rules of differentiation, $f^{(n)}$ exists and is continuous for $x \neq 0$ and for each $n \in \mathbf{N}$. The only question then is whether these higher derivatives exist at 0 and whether $f^{(n)}$ is continuous at 0.

We note first that because of Leibnitz's Rule (Section 4.2, Exercise 5), the formula for $f^{(n)}(x)$ for $x \neq 0$ will be a linear combination of terms of the form $x^{-n}\, e^{-1/x^2}$. This can be shown formally by induction but it is also clear by trying a few examples. By remarks made in part (a) above it follows that $\lim_{x\to 0} x^{-n}\, e^{-1/x^2} = 0$ for all $n \in \mathbf{N}$ and hence that $\lim_{x\to 0} f^{(n)}(x) = 0$ for all $n \in \mathbf{N}$.

It remains to show that for each $n \in \mathbf{N}$, $f^{(n)}(0) = 0$. To see this we can make an informal induction argument. In part (a) we proved the result for $n = 1$. Note that $f^{(n)}(0) = \lim_{h\to 0}(f^{(n-1)}(h) - f^{(n-1)}(0))/h$ and that by induction we can assume that $f^{(n-1)}(0) = 0$ so that our limit becomes $\lim_{h\to 0} f^{(n-1)}(h)/h$. Again by induction we know that $f^{(n-1)}(h)$ consists of a linear combination of terms of the form $h^{-n}\,e^{-1/h^2}$ and so $f^{(n-1)}(h)/h$ will consist of the same linear combination of terms of the form $h^{-n-1}\,e^{-1/h^2}$. Since the limit of all such terms is zero, it follows that $f^{(n)}(0) = \lim_{h\to 0}(f^{(n-1)}(h) - f^{(n-1)}(0))/h = 0$ and $f^{(n)}(x)$ is continuous on $\mathbf{R}$.

Exercise 5.

The proof of each of these results is a direct application of the Mean Value Theorem.

(a). Let $x \in \mathbf{R}$ be fixed. By the MVT there is a number c between x and 0 such that $f(x) - f(0) = f'(c)(x - 0) = 0$ since $f'(c) = 0$ for all $c \in \mathbf{R}$. Therefore, $f(x) = f(0)$ for all $x \in \mathbf{R}$ as desired.

(b). Let $x \in \mathbf{R}$. Then as above there is a c between x and 0 such that $f(x) - f(0) = f'(c)(x - 0)$. Since $f(0) = 1$, $f(x) = f'(c)x + 1$. Since $|f'(c)| \leq 1$, $|f(x)| \leq |f'(c)||x| + 1 \leq |x| + 1$ as desired.

(c). By the MVT there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. Since $f'(c) \geq 0$ and since $b - a > 0$, $f(b) - f(a) = f'(c)(b - a) \geq 0$. Therefore $f(b) \geq f(a)$ as desired.