MATH 315 – HOMEWORK 7
SOLUTIONS TO SELECTED EXERCISES

Section 3.3, Exercise 1.

(a). Since both e^x and x^2 are continuous, we can use the Intermediate Value Theorem to show the existence of a root by finding real numbers a and b such that $f(a) = e^a - a^2 < 0$ and $f(b) = e^b - b^2 > 0$. Then the IVT says there is an $x \in (a, b)$ such that $f(x) = 0$. By definition, $e^0 = 1$ so that $f(0) = e^0 - 0^2 = 1 > 0$ so that $f(0) > 0$. Since $e > 2$, $e^{-1} < 2^{-1} < 1 = (-1)^2$ so that $f(-1) < 0$. Hence there is an $x \in (-1, 0)$ such that $e^x = x^2$.

Exercise 4.

If $f(a) < M$, let $\epsilon = M - f(a) > 0$ and choose $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Then for all such x,

$$\begin{aligned} f(x) &= f(x) - f(a) + f(a) < |f(x) - f(a)| + f(a) \\ &< M - f(a) + f(a) = M. \end{aligned}$$

Hence $f(x) < M$ for all $x \in (a - \delta, a + \delta)$.

Exercise 10.

Let M be an arbitrary number in the range of f, that is, let M be such that for some $x_0 \in \mathbf{R}$, $f(x_0) = M$. Since $\lim_{x\to\infty} f(x) = \infty$ there is a number $b > 0$ such that if $x > b$ then $f(x) > M$. (We know such a b exists and we can ensure that $b > 0$ since if a certain b works then any larger b will also work.) Similarly, since $\lim_{x\to-\infty} f(x) = \infty$ we can find a number $a < 0$ such that $f(a) > M$ and if $x \le a$ then $f(x) > M$. Note that since $f(x_0) = M$, $x_0 \in [a, b]$ since any $x \notin [a, b]$ must satisfy $f(x) > M$. Since f is continuous on $\mathbf{R}$ it is continuous on the interval $[a, b]$ and there exists $x_m \in [a, b]$ such that $f(x_m) = \inf_{x\in[a,b]} f(x)$. To see that in fact $f(x_m) = \inf_{x\in\mathbf{R}} f(x)$ note that if $x \notin [a, b]$ then $f(x) > M = f(x_0) \ge f(x_m)$ since $x_0 \in [a, b]$. Therefore for any $x \in \mathbf{R}$, $f(x) \ge f(x_m)$.

Section 3.4, Exercise 4.

(a). Let $\epsilon > 0$ and choose B so large that if $x > B$ then $|f(x) - L| < \epsilon/2$. Since $[0, B+1]$ is a closed bounded interval and since f is continuous on $[0, B+1]$ it is uniformly continuous there. Hence there is a $0 < \delta < 1$ such that if $x, y \in [0, B+1]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/2$. Now for any $x, y \in [0, \infty)$, assume that $|x - y| < \delta$. If $x, y \in [0, B+1]$, we have $|f(x) - f(y)| < \epsilon/2 < \epsilon$. If $x, y \notin [0, B]$, then $|f(x) - f(y)| \leq |f(x) - L| + |f(y) - L| < \epsilon/2 + \epsilon/2 = \epsilon$. Finally if $x \in [0, B]$ and $y \notin [0, B+1]$ (or $y \in [0, B]$ and $x \notin [0, B+1]$) then $|x - y| \geq \delta$ since $\delta < 1$, so this case need not be considered.

(b). Let $f(x) = 1/(x^2 + 1)$. Since $\lim_{x\to\infty} f(x) = 0$, part (a) implies that f is uniformly continuous on $[0, \infty)$. Since $\lim_{x\to-\infty} f(x) = 0$, an argument almost identical to that in part (a) implies that f is uniformly continuous on $(-\infty, 0]$. Since $f(x)$ is continuous on $\mathbf{R}$, it is uniformly continuous on $[-1, 1]$. Now let $\epsilon > 0$ and choose $0 < \delta < 1$ so that if $x, y \in [0, \infty)$, $x, y \in (-\infty, 0]$, or $x, y \in [-1, 1]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Now for any $x, y \in \mathbf{R}$ let $|x - y| < \delta$. If $x, y \in [0, \infty)$ or $x, y \in (-\infty, 0]$ then we are done. If $x < 0$ and $y \geq 0$ (or $y < 0$ and $x \geq 0$) then since $\delta < 1$ and $|x - y| < \delta$, $x, y \in [-1, 1]$, and we are done.

Exercise 5.

(a). Suppose that f is not bounded on I. Let $x_1 \in I$ be such that $f(x_1) > 1$ and for each $n \in \mathbf{N}$, $n > 1$ we can find $x_n \in I$ such that $f(x_n) > f(x_{n-1}) + 1$. This means that for $n > m$,

$$f(x_n) - f(x_m) = f(x_n) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + \cdots + f(x_{m+1}) - f(x_m) > n - m.$$

By adding the endpoints to I if necessary we can make I a closed, bounded interval and by B-W we can find a convergent subsequence of $\{x_n\}$, say $\{x_{n_k}\}$. We will show that f is not uniformly continuous on I. Let $\epsilon = 1$ and choose $\delta > 0$. Since $\{x_{n_k}\}$ is convergent, it is also Cauchy and so we can find k and $l \in \mathbf{N}$ with $k > l$ such that $|x_{n_k} - x_{n_l}| < \delta$. But by construction $f(x_{n_k}) - f(x_{n_l}) > n_k - n_l \geq 1$. Hence f is not uniformly continuous on I.

(b). If $f(x) = x$ and $I = [0, \infty)$ then f is uniformly continuous but unbounded on I. If $f(x) = 1/x$ on $I = (0, 1)$ then f is (merely) continuous but unbounded on I.