

MATH 315 – HOMEWORK 6  
SOLUTIONS TO SELECTED EXERCISES

**Section 3.1, Exercise 1.**

(a). Let  $\epsilon > 0$  and let  $0 < \delta < \min(1, \epsilon/4)$ . Note that

$$|f(x) - L| = |x^2 - x - 2| = |x + 1||x - 2|.$$

Since  $0 < \delta < 1$ ,  $|x - 2| < \delta$  implies that  $|x - 2| < 1$  which implies that  $1 < x < 3$  which implies that  $|x + 1| < 4$ . Since also  $\delta < \epsilon/4$ ,  $0 < |x - 2| < \delta$  implies that

$$|f(x) - L| = |x + 1||x - 2| < 4\delta < 4(\epsilon/4) = \epsilon.$$

**Exercise 6.**

Suppose to the contrary that  $L = \lim_{x \rightarrow a} f(x) > \lim_{x \rightarrow a} g(x) = M$ . Then  $L - M > 0$  and there is a  $\delta > 0$  such that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < (L - M)/2$  and  $|g(x) - M| < (L - M)/2$ . Therefore for all such  $x$ ,

$$\begin{aligned} g(x) - f(x) &= (g(x) - M) + (M - L) + (L - f(x)) \\ &\leq |g(x) - M| + (M - L) + |L - f(x)| \\ &< (L - M)/2 + M - L + (L - M)/2 = 0. \end{aligned}$$

Hence for all such  $x$ ,  $f(x) > g(x)$  contradicting the hypothesis of Theorem 3.10.

**Exercise 8.**

(a). Let  $\epsilon > 0$  and choose  $\delta > 0$  such that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ . By the triangle inequality and for the same  $\delta > 0$  as above, if  $0 < |x - a| < \delta$  then  $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$ .

(b). Let  $f(x) = x/|x|$  for  $x \neq 0$ . Then  $|f(x)| = 1$  for all  $x \neq 0$  and so  $\lim_{x \rightarrow 0} |f(x)| = 1$ . However since  $f(x) = 1$  for  $x > 0$  and  $-1$  for  $x < 0$  we have that  $\lim_{x \rightarrow 0^+} f(x) = 1$  and  $\lim_{x \rightarrow 0^-} f(x) = -1$ . Hence  $\lim_{x \rightarrow 0} f(x)$  does not exist.

### Section 3.2, Exercise 1.

(a). Given  $\epsilon > 0$  let  $\delta > 0$  be arbitrary. If  $0 < x < \delta$  then  $|x| = x$  so that  $|x|/x = 1$  and  $|f(x) - 1| = |(|x|/x) - 1| = |1 - 1| = 0 < \epsilon$ .

(b). Let  $M \in \mathbf{R}$ . If  $M \geq 0$  then let  $\delta > 0$  be arbitrary. If  $0 < x < \delta$  then  $-1/x < 0 \leq M$ . If  $M < 0$  let  $0 < \delta < -1/M$ . If  $0 < x < \delta < -1/M$  then  $x < -1/M$  which implies that  $-1/x < M$  as desired.

(c). Note first that if  $x \neq 1$  then

$$\frac{x-1}{x^2+x-2} = \frac{x-1}{(x-1)(x+2)} = \frac{1}{x+2}.$$

Given  $M > 0$  let  $0 < \delta < \min(3, 1/M)$ . Then if  $-2 < x < -2 + \delta$ ,  $x - 1 < -3 + \delta < -3 + 3 = 0$  so that  $x \neq 1$ , and  $0 < x + 2 < \delta < 1/M$  which implies that  $(x-1)/(x^2+x-2) = 1/(x+2) > M$ .

(d). Given  $M > 0$  let  $0 < \delta < \min(1, 1/(3M))$ . If  $1 < x < 1 + \delta$  then  $0 < x - 1 < \delta < 1/3M$  which implies that  $1/(x-1) > 3M$ . If  $1 < x < 1 + \delta$  then  $2 < x + 1 < 2 + \delta < 3$  or  $1/3 < 1/(x+1)$ . Therefore if  $1 < x < 1 + \delta$  then

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} > \frac{1}{3(x-1)} > \frac{3M}{3} = M.$$

### Exercise 9.

( $\Leftarrow$ ) If  $f(x) = 0$  for all  $x \in [0, 1]$  then in particular  $f(x) = 0$  for all  $x \in \mathbf{Q} \cap [0, 1]$ .

( $\Rightarrow$ ) Suppose that  $f(q) = 0$  for all  $q \in \mathbf{Q} \cap [0, 1]$ , and let  $x \in [0, 1]$ . If  $x = 0$  or  $1$  then  $x \in \mathbf{Q}$  so  $f(x) = 0$ . Hence we can assume that  $x \in (0, 1)$ . By Problem 5, Section 2.2 there is a sequence  $\{r_n\} \subseteq \mathbf{Q}$  such that  $r_n \rightarrow x$ . Since  $x \in (0, 1)$  then for all large  $n$ ,  $r_n \in (0, 1)$  and hence  $f(r_n) = 0$ . Therefore by Theorem 3.6 and the hypothesis of the theorem,  $f(x) = \lim_{r \rightarrow x} f(r) = \lim_{n \rightarrow \infty} f(r_n) = 0$  as required.