

MATH 315 – HOMEWORK 5
SOLUTIONS TO SELECTED EXERCISES

Section 2.3, Exercise 2.

Choose $x_1 \in E$ arbitrarily. Since $\sup E \notin E$, $x_1 < \sup E$. Hence by the Approximation Theorem there is an $x_2 \in E$ such that $x_1 < x_2 < \sup E$, and such that $\sup E - (1/2) < x_2 < \sup E$. In other words, we have chosen $x_2 \in E$ satisfying $\sup E - \alpha < x_2 < \sup E$ where $\alpha = \min(1/2, \sup E - x_1) > 0$. Continuing in this fashion, we can choose for each $n \in \mathbf{N}$, $n > 2$, a number $x_n \in E$ such that $x_{n-1} < x_n < \sup E$ and $\sup E - 1/n < x_n < \sup E$. By construction $\{x_n\}$ is strictly increasing and since $|x_n - \sup E| < 1/n$, $x_n \rightarrow \sup E$.

Exercise 4.

In previous exercises, we have shown that $\{x_n\}$ is a decreasing sequence, $x_n > 0$ for all $n \in \mathbf{N}$ (Exercise 4(a), Section 1.2), and that it converges to either 0 or 1 (Exercise 7(b), Section 2.2). Since $\{x_n\}$ is decreasing, $x_n \leq x_1 < 1$ for all $n \in \mathbf{N}$ and by the Comparison Theorem, $x = \lim x_n \leq x_1 < 1$. Hence $x = 0$.

Moreover, since $x_n > 0$, we can compute

$$\frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n} \cdot \frac{1 + \sqrt{1 - x_n}}{1 + \sqrt{1 - x_n}} = \frac{1 - (1 - x_n)}{x_n(1 + \sqrt{1 - x_n})} = \frac{1}{1 + \sqrt{1 - x_n}}.$$

Therefore, combining Theorem 2.12 and Exercise 4, Section 2.2,

$$\lim \frac{x_{n+1}}{x_n} = \lim \frac{1}{1 + \sqrt{1 - x_n}} = \frac{1}{1 + \sqrt{1 - \lim x_n}} = \frac{1}{1 + \sqrt{1 - 0}} = 1/2.$$

Exercise 8.

Note first that if $x_0 = 1$, then $x_n = 1$ for all $n \in \mathbf{N}$ and $\lim x_n = 1$. If $x_0 < 1$ then $x_1 = 1/2 + x_0/2 < 1/2 + 1/2 = 1$, and a simple induction argument shows that $x_n < 1$ for all $n \in \mathbf{N}$ (you should fill in the details). Since $x_n < 1$, $x_n = x_n/2 + x_n/2 < 1/2 + x_n/2 = x_{n+1}$, and $\{x_n\}$ is strictly increasing. Since also $\{x_n\}$ is bounded above (by 1), it converges say to x . If $x_0 > 1$ then $1 = 1/2 + 1/2 < 1/2 + x_0/2 = x_1$ and a simple induction argument shows that $x_n > 1$ for all $n \in \mathbf{N}$. Since $x_n > 1$, $x_n = x_n/2 + x_n/2 > 1/2 + x_n/2 = x_{n+1}$, and $\{x_n\}$ is strictly decreasing. Since also $\{x_n\}$ is bounded below (by 1), it converges say to x .

In either case, taking the limit of both sides of $x_n = 1/2 + x_{n-1}/2$ gives $x = 1/2 + x/2$ which implies $x = 1$.

Section 2.4, Exercise 3.

We will use the fact that if $x, y \in \mathbf{N}$ are distinct, then $|x - y| \geq 1$. Let $\epsilon = 1$ and choose N so that if $n, m \geq N$ then $|x_m - x_n| < \epsilon = 1$. Let $a = x_N$. We will show that $x_n = a$ for all $n \geq N$. If $n \geq N$ and $x_n \neq a$ then since $x_N, a \in \mathbf{N}$, $1 \leq |x_n - a| = |x_n - x_N| < 1$, a contradiction. Hence $x_n = a$ for all $n \geq N$.

Exercise 6.

We will use Exercise 4. Consider the partial sum $\sum_{k=n}^m (-1)^k/k$. Suppose first that n is even. Then if m is also even,

$$\begin{aligned} \sum_{k=n}^m (-1)^k/k &= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots - \frac{1}{m-1} + \frac{1}{m} \\ &= \frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \cdots - \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &< \frac{1}{n} \end{aligned}$$

since $1/k - 1/(k+1) > 0$ for all $k \in \mathbf{N}$. If m is odd then we have the same upper bound since the last term in the partial sum would be negative. If m is odd, then

$$\begin{aligned} \sum_{k=n}^m (-1)^k/k &= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots + \frac{1}{m-1} - \frac{1}{m} \\ &= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &> 0 \end{aligned}$$

for the same reason as above. Also, if m is even then the same lower bound holds since the last term in the partial sum would be positive.

If n is odd then the first term in the partial sum is negative so

$$\sum_{k=n}^m (-1)^k/k < \sum_{k=n+1}^m (-1)^k/k < \frac{1}{n+1} < \frac{1}{n}$$

by the same argument as above, and similarly,

$$\sum_{k=n}^m (-1)^k/k = \sum_{k=n-1}^m (-1)^k/k - \frac{1}{n-1} > \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n-1} = -\frac{1}{n}.$$

Putting everything together we have in any case that $-\frac{1}{n} < \sum_{k=n}^m (-1)^k/k < \frac{1}{n}$ or

that $\left| \sum_{k=n}^m (-1)^k/k \right| < \frac{1}{n}$. From this the criteria of Exercise 4 follows (fill in the details here if you like).