

MATH 315 – HOMEWORK 4
SOLUTIONS TO SELECTED EXERCISES

Section 2.1, Exercise 5.

(b). By part (a) of this problem, there is a number C such that for all $n \in \mathbf{N}$, $|x_n| \leq C$. Also note that since $n \geq 1$ then $n^k \geq 1$ for all $k \in \mathbf{N}$ and $n \in \mathbf{N}$. This implies that for all $k \in \mathbf{N}$ and $n \in \mathbf{N}$, $(1/n^k) \leq (1/n)$. Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ so that $N > C/\epsilon$. Hence $n \geq N$ implies that $n > C/\epsilon$ or $C/n < \epsilon$. Therefore, $|x_n/n^k| = |x_n|/n^k \leq |x_n|/n \leq C/n < \epsilon$.

Exercise 6.

(a). Suppose that $x_n \rightarrow a$ and $y_n \rightarrow a$ and let $\epsilon > 0$. Choose $N_1 \in \mathbf{N}$ so that if $n \geq N_1$ then $|x_n - a| < \epsilon/2$ and $N_2 \in \mathbf{N}$ so that if $n \geq N_2$ then $|y_n - a| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then using the triangle inequality, $|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |y_n - a| < \epsilon/2 + \epsilon/2 = \epsilon$.

(b). Since every convergent sequence is bounded and since $\{n\}_{n=1}^{\infty}$ is not bounded, it cannot be convergent.

(c). The converse of part (a) says that *If $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n - y_n \rightarrow 0$, then $\{x_n\}$ and $\{y_n\}$ converge to the same point.* To see that this is false, let $x_n = y_n = n$. By part (b) of this problem neither $\{x_n\}$ nor $\{y_n\}$ converges, but $x_n - y_n = 0 \rightarrow 0$ as $n \rightarrow \infty$.

Section 2.3, Exercise 4.

(a). Assume first that $x > 0$. Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ so that if $n \geq N$ then $|x_n - x| < \epsilon\sqrt{x}$. For such an n , using the fact that $x_n \geq 0$,

$$|\sqrt{x_n} - \sqrt{x}| = \left| (\sqrt{x_n} - \sqrt{x}) \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < (\epsilon\sqrt{x})/\sqrt{x} = \epsilon.$$

Assume now that $x = 0$, and let $\epsilon > 0$. Choose $N \in \mathbf{N}$ so that if $n \geq N$ then $|x_n| < \epsilon^2$. By (8) on p. 7, $0 \leq |x_n| = x_n < \epsilon^2$ implies that $0 \leq \sqrt{x_n} < \epsilon$. Hence $\sqrt{x_n} \rightarrow \sqrt{0} = 0$.

Exercise 7.

(a). Note first that since square roots are always positive and since $x_1 \geq 0$ that $x_n \geq 0$ for all $n \in \mathbf{N}$. Therefore $\lim x_n = x \geq 0$. Taking the limit of both sides of the equation $x_{n+1} = \sqrt{2 + x_n}$ gives the equation $x = \sqrt{2 + x}$. Squaring both sides gives $x^2 = 2 + x$ and manipulating gives $x^2 - x - 2 = 0$. Solving this quadratic gives $x = -1$ or $x = 2$. Since $x \geq 0$, it follows that $x = 2$.

(b). $x_{n+1} = 1 - \sqrt{1 - x_n}$ implies that $1 - x_{n+1} = \sqrt{1 - x_n}$. Taking the limit of both sides of this equation gives $1 - x = \sqrt{1 - x}$. The only solutions to this equation are $1 - x = 0$ or $1 - x = 1$, that is, $x = 0$ or 1 .