MATH 315 – HOMEWORK 3
SOLUTIONS TO SELECTED EXERCISES

Section 1.3, Exercise 5.

(a). Let $t = \inf E$ and suppose that there is no $a \in E$ such that $a < t + \epsilon$. This means that for all $a \in E$, $a \geq t + \epsilon$, that is, that $t' = t + \epsilon$ is a lower bound of E. By the definition of infimum, $t' \leq t$, that is, $t + \epsilon \leq t$ which implies that $\epsilon \leq 0$. (Note that this is a proof by contrapositive.) Hence such an $a \in E$ exists and by definition, $\inf E \leq a$ also.

(b). Let $\epsilon > 0$. By the approximation property for suprema there is a $b \in -E$ such that $b > \sup(-E) - \epsilon$. Multiplying this inequality by -1 gives $-b < -\sup(-E) + \epsilon$. By Theorem 1.28 (ii), $-\sup(-E) = \inf E$ and by definition, $a = -b \in E$, and this $a \in E$ satisfies $a < \inf E + \epsilon$.

Exercise 8.

For each $n \in \mathbf{N}$, define $E_n = \{x_k : k \geq n\}$. Then $E_n \neq \emptyset$ and since $x_k \leq M$ for all $k \in \mathbf{N}$, E_n is bounded above. Hence $s_n = \sup E_n$ exists and is finite. Since $E_{n+1} \subseteq E_n$, Theorem 1.29 implies that $s_{n+1} \leq s_n$ for all $n \in \mathbf{N}$.

The analogous result for infima uses the fact that since $-M \leq x_k$ for all $k \in \mathbf{N}$, E_n is bounded below and hence $t_n = \inf E_n$ exists and is finite, and Theorem 1.29 implies that $t_{n+1} \geq t_n$ for all $n \in \mathbf{N}$.