MATH 315 – HOMEWORK 2
SOLUTIONS TO SELECTED EXERCISES

Section 1.2, Exercise 2.

(b). If $n = 1$ then the inequality to be proved reduces to $a + b \geq a + b$ which is true. If $n \geq 2$ then

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k} \\ &= \sum_{k=0}^{n-2} \binom{n}{k} a^k\, b^{n-k} + \binom{n}{n-1} a^{n-1}\, b + \binom{n}{n} a^n \\ &= na^{n-1}b + a^n + \sum_{k=0}^{n-2} \binom{n}{k} a^k\, b^{n-k} \\ &\geq na^{n-1}b + a^n\end{aligned}$$

since $a \geq 0$ and $b \geq 0$ implies that for all $k \in \mathbf{N}$, $a^k \geq 0$ and $b^k \geq 0$.

(c). If $n = 1$ the inequality reduces to $2 \geq 2$. If $n \geq 2$ then

$$\begin{aligned}(1+1/n)^n &= \sum_{k=0}^{n} \binom{n}{k} (1/n)^k\, 1^{n-k} \\ &= \binom{n}{0} (1/n)^0 + \binom{n}{1} (1/n)^1 + \sum_{k=2}^{n} \binom{n}{k} (1/n)^k \\ &\geq 1 + n(1/n) = 2\end{aligned}$$

since for all $k \in \mathbf{N}$ and $n \in \mathbf{N}$, $(1/n)^k \geq 0$.

Exercise 4.

(a). We will prove this by induction. Since $0 < x_1 < 1$ and since $x_2 = 1-\sqrt{1-x_1}$ Section 1.1, Exercise 5(a) says that $0 < x_2 < x_1 < 1$ and the result holds for $n = 1$. Suppose that for $n \in \mathbf{N}$, $0 < x_{n+1} < x_n < 1$. Then in particular, $0 < x_{n+1} < 1$ and since $x_{n+2} = 1-\sqrt{1-x_{n+1}}$ it follows from Section 1.1, Exercise 5(a) that $0 < x_{n+2} < x_{n+1} < 1$. Hence for all $n \in \mathbf{N}$, $0 < x_{n+1} < x_n < 1$.

Exercise 9.

We will prove that for all $n \in \mathbf{N}$, $2^{2n-1} + 3^{2n-1}$ is a multiple of 5 by induction. If $n = 1$ then the assertion reduces to $2+3$ is a multiple of 5 which is true. Suppose that the result holds for $n \in \mathbf{N}$ and consider $2^{2(n+1)-1} + 3^{2(n+1)-1}$.

$$\begin{aligned}2^{2(n+1)-1} + 3^{2(n+1)-1} &= 2^{(2n-1)+2} + 3^{(2n-1)+2} \\ &= 4\, 2^{2n-1} + 9\, 3^{2n-1} \\ &= 4\,(2^{2n-1} + 3^{2n-1}) + 5\, 3^{2n-1}.\end{aligned}$$

By the induction hypothesis, $2^{2n-1} + 3^{2n-1}$ is a multiple of 5 and it follows that $4\,(2^{2n-1}+3^{2n-1})+5\,3^{2n-1}$ is also a multiple of 5. Hence $2^{2n-1}+3^{2n-1}$ is a multiple of 5 for all $n \in \mathbf{N}$.

Exercise 10.

(a). We wish to prove that $c_k - b_k$ is constant for all $k \in \mathbf{N}$. In particular we will show that $c_k - b_k = c_1 - b_1 = 13 - 12 = 1$ for all $k \in \mathbf{N}$, by induction. The result is clearly true for $k = 1$. Suppose that the result holds for $k \in \mathbf{N}$. Then

$$\begin{aligned} c_{k+1} - b_{k+1} &= 2a_k + c_k + 2 - 2a_k - b_k - 2 \\ &= c_k - b_k = 1 \end{aligned}$$

by the induction hypothesis.

(b). We will prove that $a_k^2 + b_k^2 = c_k^2$ by induction on k. If $k = 0$ then $a_0^2 + b_0^2 = 3^2 + 4^2 = 5^2 = c_0^2$ and the result holds in this case. Assume that the result holds for $k \in \mathbf{N}$. Then

$$\begin{aligned} a_{k+1}^2 + b_{k+1}^2 &= (a_k + 2)^2 + (2a_k + b_k + 2)^2 \\ &= 5a_k^2 + 12a_k + 4b_k + 4a_k b_k + 8 + b_k^2 \end{aligned}$$

Also, bearing in mind that by part (a), $c_k = b_k + 1$ and that by the induction hypothesis $c_k^2 = a_k^2 + b_k^2$,

$$\begin{aligned} c_{k+1}^2 &= (2a_k + c_k + 2)^2 \\ &= 4a_k^2 + 4a_k c_k + 8a_k + c_k^2 + 4c_k + 4 \\ &= 4a_k^2 + 4a_k(b_k + 1) + 8a_k + (a_k^2 + b_k^2) + 4(b_k + 1) + 4 \\ &= 5a_k^2 + 4a_k b_k + 12a_k + b_k^2 + 4b_k + 8 \\ &= a_{k+1}^2 + b_{k+1}^2 \end{aligned}$$

by above. Hence the result holds for all $k \in \mathbf{N}$.