MATH 315 – HOMEWORK 1
SOLUTIONS TO SELECTED EXERCISES

Section 1.1, Exercise 4.

Proof of (7). Suppose that $0 \le a < b$ and $0 \le c < d$. If $c > 0$, the multiplicative property implies that $ac < bc < bd$ so that $ac < bd$ by the transitive property. If $c = 0$ then since $b > 0$ and $d > 0$, $0 = ac < bd$ by the multiplicative property.

Proof of (8). Suppose that $0 \le a < b$. Then by (7) with $a = c$ and $b = d$, $0 \le a^2 < b^2$. For the second inequality note first that if $a \ge 0$ then $\sqrt{a} \ge 0$ by definition. We will prove the remaining inequality by proving its contrapositive, that is, we will assume the negation of the conclusion and prove the negation of the hypothesis. Suppose that $\sqrt{b} \le \sqrt{a}$. If $\sqrt{b} = \sqrt{a}$ then $(\sqrt{b})^2 = (\sqrt{a})^2$, that is, $b = a$, and if $\sqrt{b} < \sqrt{a}$ then by the first part of (8) proved above $(\sqrt{b})^2 < (\sqrt{a})^2$, that is $b < a$. Hence $b \le a$.

Proof of (9). First note that if $a > 0$ then $1/a > 0$ for if $a > 0$ and $1/a < 0$ then by the multiplicative property, $a \cdot (1/a) = 1 < 0$, which is absurd. Suppose that $0 < a < b$. Since $a > 0$ and $b > 0$, the argument above says that $1/a > 0$ and $1/b > 0$ and by the multiplicative property, $(1/a)(1/b) > 0$. Again applying the multiplicative property, $0 < (a)(1/a)(1/b) < (b)(1/a)(1/b)$, that is, $0 < 1/b < 1/a$.

Exercise 5.

(a). If $0 < a < 1$ then by the multiplicative and additive properties of the "less than" relation, $-1 < -a < 0$ and $0 < 1 - a < 1$. By (8), $0 < \sqrt{1-a} < 1$, which implies $1 - \sqrt{1-a} > 0$, and by (6), $0 < 1 - a < \sqrt{1-a}$, which implies $1 - \sqrt{1-a} < a$ as desired.

(b). Since $a > 2$, $a - 1 > 1$ and by (8), $\sqrt{a-1} > 1$ which implies $2 < 1 + \sqrt{a-1}$. By (6), $a - 1 > \sqrt{a-1}$ which implies $1 + \sqrt{a-1} < a$ as desired.

(c). Suppose that $0 \le a \le b$. By (8), $0 \le \sqrt{a} \le \sqrt{b}$ and from the multiplicative property of the "less than" relation it follows that $a = \sqrt{a}\,\sqrt{a} \le \sqrt{a}\,\sqrt{b} = \sqrt{ab}$, that is, $a \le G(a, b)$.

By (5), $0 \le (\sqrt{b} - \sqrt{a})^2 = b - 2\sqrt{ab} + a$. This implies that $2\sqrt{ab} \le a + b$ or equivalently $\sqrt{ab} \le (a + b)/2$, that is, $G(a, b) \le A(a, b)$.

Finally, by the additive property of the "less than" relation, $a \le b$ implies that $(a + b)/2 \le (b + b)/2 = b$, or equivalently $A(a, b) \le b$.

To prove the last part of the problem, if $G(a, b) = A(a, b)$ then $2\sqrt{ab} = a + b$ and consequently $a + b - 2\sqrt{ab} = (\sqrt{b} - \sqrt{a})^2 = 0$. By (5), $\sqrt{b} - \sqrt{a} = 0$ and $b = a$. If $0 \le a = b$ then $A(a, b) = (a+b)/2 = (a+a)/2 = a$ and $G(a, b) = \sqrt{ab} = \sqrt{a\,a} = a$, so $A(a, b) = G(a, b)$.

Exercise 9.

Let $a_1, a_2, b_1, b_2 \in \mathbf{R}$. By (5), $0 \leq (a_1b_2 - a_2b_1)^2 = a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_2a_2b_1$ or equivalently $2a_1b_2a_2b_1 \leq a_1^2b_2^2 + a_2^2b_1^2$. Adding $a_1^2b_1^2 + a_2^2b_2^2$ to both sides gives $a_1^2b_1^2 + 2a_1b_2a_2b_1 + a_2^2b_2^2 \leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$. Manipulating this inequality gives $(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)^2\,(b_1^2 + b_2^2)^2$.

Exercise 10.

(a). Assuming $|x - a| < \epsilon$ for some $\epsilon > 0$, the triangle inequality implies that $|x| - |a| \leq |x - a| < \epsilon$ and hence that $|x| < |a| + \epsilon$. Therefore,

$$\begin{aligned}
|xy - ab| &= |xy - xb + xb - ab| \\
&\leq |xy - xb| + |xb - ab| \\
&= |x|\,|y - b| + |b|\,|x - a| \\
&< (|a| + \epsilon)\,\epsilon + |b|\,\epsilon \\
&= (|a| + |b|)\epsilon + \epsilon^2
\end{aligned}$$

as desired.

(b). From the triangle inequality we have as above that $|y| < |b| + \epsilon$ and applying part (a) with $b = a$ and $x = y$ to $|x^2 - a^2|$ it follows that $|x^2 - a^2| < 2|a|\epsilon + \epsilon^2$. Therefore,

$$\begin{aligned}
|x^2y - a^2b| &= |x^2y - a^2y + a^2y - a^2b| \\
&\leq |x^2y - a^2y| + |a^2y - a^2b| \\
&= |y|\,|x^2 - a^2| + |a^2|\,|y - b| \\
&< (|b| + \epsilon)\,(2|a|\epsilon + \epsilon^2) + |a|^2\,\epsilon \\
&= 2|ab|\epsilon + |b|\,\epsilon^2 + 2|a|\epsilon^2 + \epsilon^3 + |a|^2\,\epsilon \\
&= \epsilon(2|ab| + |a|^2) + \epsilon^2(|b| + 2|a|) + \epsilon^3
\end{aligned}$$

as desired.