# MATH 315 – FALL 2008 – MIDTERM EXAM – SOLUTIONS

1. (a) (5 pts.) Give the definition of a *finite set* and of a *countably infinite set.*

   (b) (10 pts.) Show that if $E \subseteq \mathbf{R}$ is a bounded finite set then $\sup E \in E$. (Hint: You may want to use the Approximation Property for suprema.)

   (c) (5 pts.) Give an example of an infinite set in $\mathbf{R}$, bounded above, that does not contain its supremum. (Hint: It is enough here to define the set and indicate its supremum. It is not necessary to prove that the supremum of the set is what you say it is.)

**Solution:**

(b). Let $s = \sup E$ and suppose that $s \notin E$. Then we know from a homework problem that there is a strictly increasing sequence $\{x_n\}$ with $x_n \in E$ for all $n \in \mathbf{N}$ such that $x_n \to s$. Since $\{x_n\}$ is strictly increasing, each element of the sequence is distinct. This means that the function $g\colon \mathbf{N} \to \{x_n\}$ given by $g(n) = x_n$ is a one-to-one function from $\mathbf{N}$ to a subset of $E$. Therefore, $E$ is at least countably infinite, contradicting the hypothesis that $E$ is finite.

Or the proof could proceed by induction on the number of elements in the set. Suppose that $E$ is nonempty and finite and that the bijection $k \mapsto x_k$ maps the set $\{1, 2, \dots, n\}$ onto $E$. If $n = 1$ then $E = \{x_1\}$ and $\sup E = x_1 \in E$. Suppose now that the hypothesis is true for $n$, and that $E = \{x_1, x_2, \dots, x_n, x_{n+1}\}$. Let $E' = \{x_1, x_2, \dots, x_n\}$. If $x_{n+1} \geq \sup E'$, then $\sup E = x_{n+1} \in E$, and if $x_{n+1} < \sup E'$ then $\sup E = \sup E' \in E' \subseteq E$ by the induction hypothesis.

(c). Let $E = (0, 1)$. Then $\sup E = 1$ but $1 \notin E$. Or let $E = \{1 - (1/n)\colon n \in \mathbf{N}\}$. Then again $\sup E = 1$ but $1 \notin E$.

2. (a) (5 pts.) Give the definition of a *convergent sequence.*

   (b) (10 pts.) Show that every convergent sequence is bounded.

**Solution:**

(b). Suppose that $x_n \to a$. Let $\epsilon = 1$ and let $N \in \mathbf{N}$ be such that if $n \geq N$ then $|x_n - a| < 1$. By the triangle inequality, $|x_n| - |a| \leq |x_n - a| < 1$ so that $|x_n| < |a| + 1$ for all $n \geq N$. Therefore, for all $n \in \mathbf{N}$, $|x_n| \leq \max(|x_1|, |x_2|, \dots, |x_{N-1}|, |a| + 1)$.

3. (a) (10 pts.) Show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences with the property that $x_n \leq y_n$ for all $n \in \mathbf{N}$, then $\lim x_n \leq \lim y_n$.

(b) (5 pts.) Give an example of convergent sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n < y_n$ for all $n \in \mathbf{N}$ but $\lim x_n \not< \lim y_n$.

**Solution:**

(a). Let $\lim x_n = a$ and $\lim y_n = b$ and suppose to the contrary that $a > b$. Let $\epsilon = a - b > 0$ and let $N \in \mathbf{N}$ be chosen so that if $n \geq N$ then $|x_n - a| < \epsilon/2$ and $|y_n - b| < \epsilon/2$. Then $a - x_n \leq |a - x_n| = |x_n - a| < \epsilon/2$ and similarly $y_n - b < \epsilon/2$. Therefore, $x_n - y_n > a - \epsilon/2 - \epsilon/2 - b = a - b - \epsilon = 0$, contradicting the hypothesis that $x_n \leq y_n$ for all $n \in \mathbf{N}$. Hence by contrapositive, $a \leq b$.

(b). Let $x_n = 0$ and $y_n = 1/n$ for all $n \in \mathbf{N}$. Then $\lim x_n = 0$ and $\lim y_n = 0$ so that $x_n < y_n$ for all $n \in \mathbf{N}$ but $\lim x_n = \lim y_n$.

4. (10 pts.) Define the sequence $\{x_n\}$ recursively by $x_{n+1} = 3 - (2/x_n)$, for $n \in \mathbf{N}$. Prove that if $x_1 > 2$ then for all $n \in \mathbf{N}$, $2 < x_{n+1} < x_n$ and that $\lim x_n = 2$.

**Solution:**

We will use induction to prove that for all $n \in \mathbf{N}$, $2 < x_{n+1} < x_n$. If $n = 1$, note that since $x_1 > 2$, $-2/x_1 > -1$ and so $x_2 = 3 - (2/x_1) > 3 - 1 = 2$. Also $x_2 < x_1$ is equivalent to $3 - (2/x_1) < x_1$ which is equivalent to $0 < x_1^2 - 3x_1 + 2$ or $0 < (x_1 - 1)(x_1 - 2)$. This latter inequality holds since $x_1 > 2$. Hence the result holds for $n = 1$.

Now assume that the result holds for $n$, that is, that $2 < x_{n+1} < x_n$. Since $x_{n+1} > 2$, $x_{n+2} = 3 - (2/x_{n+1}) > 2$ as before. Also $x_{n+2} < x_{n+1}$ is equivalent to $0 < (x_{n+1} - 1)(x_{n+1} - 2)$ again as before and this holds since $x_{n+1} > 2$. Hence $2 < x_{n+2} < x_{n+1}$ and the result follows by induction.

Since $\{x_n\}$ is a strictly decreasing sequence bounded below, the Monotone Convergence Theorem implies that $\lim x_n$ exists, call it $L$. Taking the limit of the equation $x_{n+1} = 3 - (2/x_n)$ we have that $L = 3 - (2/L)$ which is equivalent to $L^2 - 3L + 2 = 0$ which has solutions $L = 1$ and $L = 2$. Since $x_n > 2$ for all $n$, $\lim x_n = L \geq 2$ so that $L = 2$.