9.2. Inner Product (Dot Product)

A. Definition and angle between vectors.

1. Given $\mathbf{a} = [a_1, a_2, a_3]$, and $\mathbf{b} = [b_1, b_2, b_3]$, we define the inner product or dot product of $\mathbf{a}$ and $\mathbf{b}$ by
   \[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \]
   Note that $\mathbf{a} \cdot \mathbf{b}$ is a scalar quantity.

2. If $\gamma$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (where $0 \leq \gamma \leq \pi$) then
   \[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma. \]
   This is the same as the Law of Cosines.

3. If $\mathbf{a}$ is perpendicular or orthogonal to $\mathbf{b}$, then $\gamma = \pi/2$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/2) = 0$. Also if $\mathbf{a} \cdot \mathbf{b} = 0$ then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

4. If $\mathbf{b} = \mathbf{a}$ then $\gamma = 0$ so that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \cos 0 = |\mathbf{a}|^2$. This is also obvious from the definition.

5. Since $|\cos \gamma| \leq 1$ we have that $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| |\cos \gamma| \leq |\mathbf{a}| |\mathbf{b}|$. This is the Cauchy-Schwarz (C-S) inequality.

6. From the C-S inequality follows the triangle inequality: $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. (Why is it called the triangle inequality?)

7. We also have the parallelogram law: $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$. What is a geometric interpretation of this?

B. Applications.

1. Work done by a force $\mathbf{p}$ over a displacement $\mathbf{d}$ is $W = \mathbf{p} \cdot \mathbf{d}$.

2. Find the component of a force $\mathbf{w}$ in a direction $\mathbf{u}$.

3. Orthogonal projection of $\mathbf{a}$ in the direction of $\mathbf{b}$.
   a. If $p = \mathbf{a} \cdot \mathbf{b}$ then $|p|$ is the magnitude of the projection of $\mathbf{a}$ along $\mathbf{b}$.
   b. The direction of the projection is either the same as the direction of $\mathbf{b}$ or opposite to it. The projection of $\mathbf{a}$ along (or in the direction of) $\mathbf{b}$ is $p(\mathbf{b}/|\mathbf{b}|)$.

4. Equations of lines and planes.