9.1. Vectors in 2-space and 3-space

A. Definitions
1. A vector in 2/3-space is an ordered pair/triple of real numbers of the form \( \mathbf{a} = [a_1, a_2] / \mathbf{a} = [a_1, a_2, a_3] \).
2. Two vectors \( \mathbf{a} = [a_1, a_2, a_3] \), and \( \mathbf{b} = [b_1, b_2, b_3] \) are equal, that is, \( \mathbf{a} = \mathbf{b} \), if and only if \( a_1 = b_1 \), \( a_2 = b_2 \), and \( a_3 = b_3 \).
3. A vector is represented by an arrow that has magnitude and direction.
4. Two vectors are equal if and only if they have the same magnitude and direction. (More on this later.)
5. The magnitude or norm of \( \mathbf{a} = [a_1, a_2, a_3] \) is \( |\mathbf{a}| = (a_1^2 + a_2^2 + a_3^2)^{1/2} \).
6. The direction of \( \mathbf{a} \) can be given in more than one way.
   a. If \( \mathbf{a} \) is a vector in 2-space, the direction of \( \mathbf{a} \) can be specified by an angle \( \gamma \).
   b. In general direction is specified by a unit vector, that is, a vector \( \mathbf{u} = [u_1, u_2, u_3] \) where \( (u_1^2 + u_2^2 + u_3^2)^{1/2} = |\mathbf{u}| = 1 \).

B. Vectors and Representations
1. An arrow is determined by an initial point \( P = (x_1, y_1, z_1) \) and a terminal point \( Q = (x_2, y_2, z_2) \). Then the corresponding vector \( \mathbf{a} = \overrightarrow{PQ} = [x_2 - x_1, y_2 - y_1, z_2 - z_1] \).
2. If \( P = (0, 0, 0) \) then the vector is in standard position.
3. Given a vector \( \mathbf{a} = [a_1, a_2, a_3] \), the representation of \( \mathbf{a} \) in standard position is the arrow \( \overrightarrow{OQ} \) where \( O = (0, 0, 0) \) and \( Q = (a_1, a_2, a_3) \).
4. A representation of \( \mathbf{a} \) is any arrow with initial point \( P \) and terminal point \( Q \) such that \( \mathbf{a} = \overrightarrow{PQ} \).

C. Vector Operations
1. Vector Addition.
   a. Given \( \mathbf{a} = [a_1, a_2, a_3] \) and \( \mathbf{b} = [b_1, b_2, b_3] \), the vector \( \mathbf{a} + \mathbf{b} \) is given by \( \mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \).

2. Scalar Multiplication.
   a. Let \( c \) be a real number, and \( \mathbf{a} = [a_1, a_2, a_3] \). Then \( c\mathbf{a} = [ca_1, ca_2, ca_3] \).
   b. Geometrically, \( c\mathbf{a} \) has magnitude \( |c||\mathbf{a}| \) (Why?) and the same direction as \( \mathbf{a} \) if \( c > 0 \) and the opposite direction if \( c < 0 \).
   c. Recall that the direction of \( \mathbf{a} \) is a unit vector pointing in the same direction as \( \mathbf{a} \). In fact, the direction of \( \mathbf{a} \) is the vector \( \mathbf{u} = (1/|\mathbf{a}|) \mathbf{a} = \mathbf{a}/|\mathbf{a}| \).
   d. Therefore, we see that a vector is determined by its magnitude, \( |\mathbf{a}| \) and its direction \((1/|\mathbf{a}|) \mathbf{a} \) by

   \[ \mathbf{a} = |\mathbf{a}| \frac{\mathbf{a}}{|\mathbf{a}|}. \]

3. The Standard Basis. Define \( \mathbf{i} = [1, 0, 0] \), \( \mathbf{j} = [0, 1, 0] \), and \( \mathbf{k} = [0, 0, 1] \). Then any vector \( \mathbf{a} = [a_1, a_2, a_3] \) can be written as

   \[ \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}. \]