Section 17.2. Linear Fractional Transformations.

A. Definition

1. A *linear fractional transformation* is a mapping on $\mathbf{C}$ of the form $w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$.

2. If $w = \dfrac{az+b}{cz+d}$ then $w' = \dfrac{ad-bc}{(cz+d)^2}$ so that if $ad - bc = 0$ then the corresponding LFT is constant.

3. There are four special LFTs, and The four special LFTs are

   a. $w = z + b$ (Translation)

   b. $w = az$, $|a| = 1$ (Rotation)

   c. $w = az + b$ (Linear Transformation)

   d. $w = 1/z$ (Inversion in the unit circle)

B. Properties of LFTs

1. Any LFT can be realized as a composition of one or more of the four special ones. In particular we can write

$$\frac{az+b}{cz+d} = \frac{a}{c} - \left(\frac{ad-bc}{c}\right)\left(\frac{1}{cz+d}\right).$$

2. Every LFT maps lines and circles into lines and circles.

3. Every LFT maps the extended complex plane onto itself in a one-to-one fashion. In fact, if $w = \dfrac{az+b}{cz+d}$ then $z = \dfrac{dw-b}{-cw+a}$.

4. **Theorem.** Any LFT has at most two *fixed points* in the complex plane, unless it is the identity map, $f(z) = z$ in which case all points are fixed points. If a LFT has 3 fixed points, then it must be the identity. The fixed points of a LFT $w = \dfrac{az+b}{cz+d}$ must satisfy the quadratic equation $cz^2 - (a-d)z - b = 0$.