14.2. Cauchy’s Integral Theorem.

A. Contours.

1. A simple closed path in \( \mathbb{C} \) is a curve \( z(t) \) that does not touch itself. Such a curve is sometimes called a contour. That is, \( z(t) = x(t) + iy(t) \).

2. A domain \( D \) is simply connected if every simple closed path in \( D \) encloses only points in \( D \). That is, \( D \) has no holes.

B. Cauchy’s Integral Theorem.

1. **Theorem 1.** If \( f(z) \) is analytic in a simply connected domain \( D \) and if \( C \) is a contour in \( D \) then \( \int_C f(z) \, dz = 0 \).

2. Note that Cauchy’s integral theorem is like independence of path, only with a twist.

\[
\int_C f(z) \, dz = \int_C (u + iv)(dx + idy) \\
= \int_C (u \, dx - v \, dy) + i \int_C (v \, dx - u \, dy) \\
= \int \int_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \, dA + i \int \int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \, dA = 0
\]

by the C-R equations and Green’s Theorem.

C. Deformation of Path.

1. **Theorem 2.** If \( f(z) \) is analytic in a simply connected domain \( D \) then the integral of \( f \) is independent of path provided that the paths are all contained in \( D \).

2. If the endpoints of a path \( C \) are fixed, and if we can continuously deform \( C \) to another path \( C' \) with the same endpoints, then

\[
\int_C f(z) \, dz = \int_{C'} f(z) \, dz
\]

as long as all intermediate paths between \( C \) and \( C' \) contain only points where \( f(z) \) is analytic.

3. Given \( f(z) \) analytic in \( D \) and some \( z_0 \in D \), we can define for \( z \in D \),

\[
F(z) = \int_{z_0}^{z} f(z') \, dz'
\]

where the integral is taken over any path from \( z_0 \) to \( z \) that is contained in \( D \). Then \( F'(z) = f(z) \) and in particular, \( F(z) \) is analytic in \( D \).

4. If \( D \) is a doubly connected domain with boundary curves \( C_1 \) and \( C_2 \) and if \( f \) is analytic in a domain containing \( D \) and its boundary, then

\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.
\]