10.6. Surface Integrals.

A. Integrals with respect to Surface Area.

1. The *surface area element*, dA, of a surface S given by $\mathbf{r}(u,v)$, $(u,v) \in R$ is

$$dA = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

2. If G is a scalar field in 3–space then

$$\iint_S G\, dA = \iint_R G(\mathbf{r}(u,v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

   Such an integral would represent for example the mass of a surface with density function $G(x,y,z)$.

B. Integral of a Vector Field over S (Flux).

1. Let $\mathbf{v}$ be the velocity field of a fluid and ρ the (possibly variable) density of the fluid (say in gm/cm^3). Then the field $\mathbf{F} = \rho\mathbf{v}$ has units of (for example) $gm/(cm^2 - sec)$ and is called the *flux* at the point P.

2. $\mathbf{F} \cdot \mathbf{n}$ is the component of flux normal to the surface, that is, it is the *outward flow* at P.

3. $\mathbf{F} \cdot \mathbf{n}\, dA$ is the total outward flow at P. Note that the units would be for example gm/sec.

4. The integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$ is the *total flux* through the surface S.

5. So motivated, we define the integral of the vector field $\mathbf{F}$ over the surface S by

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dA &= \iint_R \mathbf{F} \cdot \left(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right) (|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv) \\
&= \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv \\
&= \iint_S \mathbf{F} \cdot d\mathbf{N}.
\end{aligned}$$

   Note that the last line is nonstandard notation.

C. The differential form of $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$.

1. Recall that we wrote

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

   where $\mathbf{F} = [F_1, F_2, F_3]$ and where $d\mathbf{r} = [dx, dy, dz] = [dx/dt, dy/dt, dz/dt]\,dt$. What would be the corresponding differential form for surface integrals?

2. Note that in the notation given above, $d\mathbf{N} = (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$ is a vector normal to the surface S at the point $P = \mathbf{r}(u, v)$ with magnitude $dA = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$.

3. Note further that

$$d\mathbf{N} = (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv = \left[\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\mathbf{i} + \cdots\right]du\,dv.$$

4. The $\mathbf{i}$ component of $d\mathbf{N}$ represents the area of the projection of the element of the surface corresponding to the $du\,dv$ rectangle in parameter space onto the yz-plane times the $\mathbf{i}$ component of the unit tangent vector $\mathbf{n}$. This means ultimately that if $\mathbf{N} = [N_1, N_2, N_3]$ then $dy\,dz = N_1\,du\,dv$ and similarly that $dx\,dz = N_2\,du\,dv$ and $dx\,dy = N_3\,du\,dv$.

5. Therefore,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy.$$

   This is the *differential form* of the surface integral of $\mathbf{F}$ over S.