10.1. Line Integrals.

A. Curves in Space.

1. A space curve \( C \) is given by a vector function \( \mathbf{r}(t) = [x(t), y(t), z(t)] \) and a range of parameters \( t \in [a, b] \).

2. We will always assume that \( C \) is smooth, that is, that \( \mathbf{r}'(t) \) is continuous and that \( \mathbf{r}'(t) \) never vanishes for all \( t \). This latter condition means that \( \mathbf{r}(t) \) has a well-defined unit tangent vector for all \( t \).

3. If \( \mathbf{r}(a) = \mathbf{r}(b) \) then we say that \( C \) is a closed path.

B. Integrals With Respect to Arclength.

1. If \( f \) is a scalar field and \( C \) a curve then
   \[
   \int_C f \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt.
   \]

2. Here \( ds \) is the arclength element from before, that is, \( ds = |\mathbf{r}'(t)| \, dt \) and \( ds \) represents the length of the infinitesimal piece of arc corresponding to the parameter range from \( t \) to \( t + dt \).

3. If we think of \( C \) as a wire suspended in 3-space whose linear density (measured in units of mass per unit length) at the point \( P \) is \( f(P) \). Then \( \int_C f \, ds \) is the total mass of the wire.

C. Integral of a Vector Field over \( C \) (Work).

1. Suppose that \( C \) is a curve or path in space and \( \mathbf{F} \) is a force field, then for each parameter value \( t \),

\[
\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
\]

is the component of the force in the direction tangent to \( C \) at the point \( \mathbf{r}(t) \).

2. Then \( \left( \mathbf{F}(\mathbf{r}) \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} \right) \, ds \) is the work done by the force on the arc element of length \( ds \). Hence the total work done by the force \( \mathbf{F} \) on the path \( C \) is

\[
\int_C \left( \mathbf{F}(\mathbf{r}) \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} \right) \, ds = \int_a^b \left( \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot d\mathbf{r}.
\]

3. So motivated, we define the integral of the vector field \( \mathbf{F} \) over the curve \( C \) by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
\]

4. Writing this out in components gives us another form (the differential form) for the same integral. If \( \mathbf{r}(t) = [x(t), y(t), z(t)] \) then

\[
\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \begin{bmatrix} dx \, dt & dy \, dt & dz \, dt \end{bmatrix}
\]

and if \( \mathbf{F} = [F_1, F_2, F_3] \) we have that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [F_1, F_2, F_3] \cdot \begin{bmatrix} dx \, dt & dy \, dt & dz \, dt \end{bmatrix} \, dt = \int_C (F_1 \, dx + F_2 \, dy + F_3 \, dz).
\]