Exercise 16.4.6.
Solution:

Using the transformations \( \sin \theta = (1/2i)(z - 1/z) \), \( \cos \theta = (1/2)(z + 1/z) \), and \( d\theta = dz/(iz) \), we arrive at

\[
\int_{0}^{2\pi} \frac{\sin^2 \theta}{5 - 4\cos \theta} d\theta = \int_{|z|=1} \frac{(-1/4)(z - 1/z)^2}{5 - 2(z + 1/z)} \frac{dz}{iz} = \frac{1}{4i} \int_{|z|=1} \frac{z^2(z - 1/z)^2}{z^2(2z^2 - 5z + 2)} dz = \frac{1}{8i} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(z - 1/2)(z - 2)} dz = \frac{2\pi i}{8i} (\text{Res}_{z=0} + \text{Res}_{z=1/2}) = \frac{\pi}{4} \left( \frac{5}{2} - \frac{3}{2} \right) = \frac{\pi}{4}
\]

where we have calculated the residues as follows.

\[
\text{Res}_{z=0} = \frac{d}{dz} \frac{(z^2 - 1)^2}{(z - 1/2)(z - 2)} \bigg|_{z=0} = \frac{2z(z^2 - 1)(z - 1/2)(z - 2) - (z^2 - 1)^2(2z - 5/2)}{(z - 1/2)^2(z - 2)^2} \bigg|_{z=0} = \frac{5/2}{(1/4)(4)} = 5/2
\]

and

\[
\text{Res}_{z=1/2} = \lim_{z \to 1/2} \frac{(z^2 - 1)^2}{z^2(z - 2)(1/2)^2(-3/2)} = -3/2.
\]

Exercise 16.4.12.
Solution:

Using the techniques described in the text, we compute

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(x^2 - 2x + 5)^2} = \lim_{R \to \infty} \left( \int_{C_1} \frac{dz}{(z - (1 + 2i))^2(z - (1 - 2i))^2} - \int_{C_2} \frac{dz}{(z - (1 + 2i))^2(z - (1 - 2i))^2} \right)
\]

where \( C_1 \) is the contour consisting of the line segment from \(-R\) to \( R\) together with the semi-circle in the upper half plane centered at 0 of radius \( R \) (counterclockwise), and \( C_2 \) is just the semi-circular part. By considerations in the text, \( \lim_{R\to\infty} \int_{C_2} z \, dz = 0 \). Also, note that only the second-order pole at \( z_0 = 1 + 2i \) is within the contour of integration. Therefore,

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2} = \lim_{R\to\infty} \int_{C_1} \frac{dz}{(z - (1 + 2i))^2(z - (1 - 2i))^2} = 2\pi i \text{ Res}_{z=1+2i} [(z - (1 + 2i))^{-2}(z - (1 - 2i))^{-2}] = 2\pi i \left. \frac{d}{dz} (z - (1 - 2i))^{-2} \right|_{z=1+2i} = 2\pi i (-2)(1 + 2i - (1 - 2i))^{-3} = 2\pi i (-i/32) = \pi/16.
\]

Exercise 17.1.14.
Solution:

First note that \( e^z = e^x \cos y + i e^x \sin y \). If \(-\pi < y \leq \pi\) then \(\cos y + i \sin y\) maps to the unit circle, and as \(x\) runs through all the real numbers, \( e^x \) runs through all strictly positive numbers. Therefore, \( e^z \) maps the strip \(-\pi < y \leq \pi\) onto the complex plane except for the origin. Finally, this implies that the strip \(-\pi < y \leq 3\pi\) maps to the complex plane except for the origin twice.

Exercise 17.2.6.
Solution:

If \( w = \frac{z + i}{z - i} \), then using the formula for the inverse we arrive at \( z = \frac{iw + i}{w - 1} \). Solving this for \( w \) we have

\[
\begin{align*}
wz - z &= iw + i \\
wz - iw &= z + i \\
w(z - i) &= z + i \\
w &= \frac{z + i}{z - i}
\end{align*}
\]

Exercise 17.2.18.
Solution:

If we let \( w = f(z) = \frac{az + b}{cz + d} \), then \( z \) is a fixed point for \( f(z) \) if and only if \( cz^2 - (a - d)z - b = 0 \). We seek values of \( a, b, c, d \) so that \( z = 0 \) is the only solution to this equation. If \( c = 0 \), then the equation is linear with solution \( z = 0 \) if and
only if \( a \neq d \) and \( b = 0 \). If \( c \neq 0 \) then in order to have \( z = 0 \) the only solution, we must have \( a = d \) so that the equation reduces to \( cz^2 = 0 \). Hence the answer is: If \( c = 0 \) then any FLT of the form \( w = (a/d)z \) with \( a \neq d \), or equivalently \( w = \alpha z \) with \( \alpha \neq 1 \) has \( 0 \) as its only fixed point, and if \( c \neq 0 \) then \( w = \frac{az}{cz + a} \) for any \( a \in \mathbb{C} \), \( a \neq 0 \) has \( z = 0 \) as its only fixed point.