Exercise 15.2.4.

Solution:

The center of this series is $z_0 = -2i$. To find the radius of convergence note that

$$\frac{|a_{n+1}|}{a_n} = \frac{(n + 1)^{n+1} n!}{(n + 1)! n^n} = \frac{(n + 1)^n (n + 1)}{n^n (n + 1)} = \left(1 + \frac{1}{n}\right)^n \to e$$

as $n \to \infty$. Therefore the radius of convergence of this series is $R = 1/e$.

Exercise 15.2.12.

Solution:

The center of this series is $z_0 = 5$. To find the radius of convergence note that

$$\frac{|a_{n+1}|}{a_n} = \frac{4^{n+1}}{(1 + i)^{n+1}} \frac{(1 + i)^n}{4^n} = \left|\frac{4}{1 + i}\right| \to \frac{4}{\sqrt{2}}$$

as $n \to \infty$. Therefore the radius of convergence of this series is $R = \sqrt{2}/4 = 1/(2\sqrt{2})$.

Exercise 15.3.2.

Solution:

(a) Direct application of the Cauchy-Hadamard Theorem gives a radius of convergence of $R = 1/4$.

(b) Note that if we let $f(z) = \sum_{n=0}^{\infty} 4^{n+1} z^n$ then taking an antiderivative gives the series $\sum_{n=0}^{\infty} \frac{4^{n+1}}{n + 1} z^{n+1}$ and taking a second antiderivative gives

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n + 1)(n + 2)} z^{n+2} = \sum_{n=1}^{\infty} \frac{4^n}{n(n + 1)} z^{n+1} = z \sum_{n=1}^{\infty} \frac{4^n}{n(n + 1)} z^n.$$

Since multiplying by $z$ does not effect the radius of convergence, the radius of convergence of the series in question is the same as that of $\sum_{n=0}^{\infty} 4^{n+1} z^n$ which is easily seen to be $R = 1/4$. 

Exercise 15.3.8.
Solution:

(a) Direct application of the Cauchy-Hadamard Theorem gives the following.
\[
\left| \frac{(2n + 2)(2n + 1) z^{2n}}{(n + 1)^{n+1}} \frac{n^n}{2n(2n - 1) z^{2n-2}} \right| = \frac{2n + 2}{2n} \frac{2n + 1}{2n - 1} \frac{n^n}{(n + 1)^{n+1}} |z|^2 \\
= \frac{2n + 2}{2n} \frac{2n + 1}{2n - 1} \left( \frac{n}{n + 1} \right)^n \frac{1}{n + 1} |z|^2 \\
\rightarrow (1)(1/e)(0)|z|^2 = 0
\]
as \(n \to \infty\). Hence the radius of convergence is \(R = \infty\).

(b) Note differentiating the series \(\sum_{n=0}^{\infty} \frac{1}{n^n} z^{2n}\) gives \(\sum_{n=1}^{\infty} \frac{2n}{n^n} z^{2n-1}\) and again gives \(\sum_{n=1}^{\infty} \frac{2n(2n - 1)}{n^n} z^{2n-2}\). Hence the radius of convergence of this series is the same as that of the first, which can be seen to be \(R = \infty\) by using the same calculation as in part (a).

Exercise 15.4.2.
Solution:

\[
\frac{1}{1 - z^3} = \sum_{n=0}^{\infty} (z^3)^n = \sum_{n=0}^{\infty} z^{3n}.
\]
This series has a radius of convergence of \(R = 1\) as can be seen in various ways including using the Cauchy-Hadamard Theorem.