Exercise 14.3.10.

Solution:
Since the circle $|z - 2i| = 4$ contains the point $2i$, and since $e^z$ is analytic on the whole complex plane, Cauchy’s Integral Formula says that

$$\int_C \frac{e^z}{z - 2i} \, dz = 2\pi i e^{2i} = 2\pi i (\cos(2) + i \sin(2)) = 2\pi (-\sin(2) + i \cos(2)).$$

Exercise 14.3.16.

Solution:
The contour consisting of $|z| = 3$ and $|z| = 1$ is the boundary of an annulus that includes the point $2i$ but not the point $0$. Therefore by Cauchy’s Integral Formula,

$$\int_C \frac{\sin z}{z^2 - 2iz} \, dz = \int_C \frac{\sin z}{z(z - 2i)} \, dz = \int_C \frac{\sin z}{z} \frac{dz}{z - 2i} = 2\pi i \frac{\sin(2i)}{2i} = \pi i \sinh(2).$$

Another way to see this is to note that if we cut the annulus whose boundary is $C$ along the real axis, we end up with two horseshoe-shaped regions, the top one includes the point $2i$ and the integrand is analytic in a simply connected domain containing the bottom one. Hence the integral over the bottom curve is zero and Cauchy’s Formula applied clearly to the top region.

Exercise 14.4.4.

Solution:
Since the circle $|z| = 2$ contains the origin, we apply Cauchy’s Derivative Formula and obtain

$$\left. \int_C \frac{\cos z}{z^{2n+1}} \, dz \right|_{z=0} = \frac{2\pi i}{(2n)!} \frac{d^{2n} \cos z}{dz^{2n}} \bigg|_{z=0} = (-1)^n \frac{2\pi i}{(2n)!}.$$

Exercise 14.4.10.

Solution:
The contour $|z| = 5$ together with $|z - 3| = 3/2$ forms the boundary of an annulus that does not contain the point $z = 4$ in its interior. Hence the function $\frac{\sin 4z}{(z - 4)^3}$ is analytic in a region containing the interior of the annulus. Hence we can apply Cauchy’s Formula to obtain

$$\int_C \frac{\sin 4z}{(z - 4)^3} \, dz = 0.$$
Exercise 15.1.16.

Solution:

We apply the ratio test to the summand and obtain:

\[
\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{10 - 15i}{10 - 15i} \right|^{n+1} \frac{n!}{(n+1)!} \frac{|10 - 15i|^n}{|10 - 15i|^n} = \frac{|10 - 15i|}{n+1} \to 0
\]

as \( n \to \infty \). Since \( 0 < 1 \) the series converges by the Ratio Test.

Exercise 15.1.22.

Solution:

We apply the ratio test to the summand and obtain:

\[
\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{1 + i}{1 + i} \right|^{n+1} \frac{(3n)!}{(3n+3)!} 1 + i} = \frac{1 + i}{n+1} \to 0
\]

as \( n \to \infty \). Since \( \frac{\sqrt{2}}{27} < 1 \) the series converges by the Ratio Test.