Exercise 13.4.4.
Solution:

In order to verify directly using the C-R Equations that \( f(z) = \frac{1}{1 - z^4} \) is analytic, we need to determine its real and imaginary parts. This is a lot of work. If we want to save ourselves some work, first realize that

\[
\frac{1}{1 - z^4} = \frac{-1}{(z - 1)(z + 1)(z - i)(z + i)}
\]

so that \( f(z) \) can be decomposed in partial fractions as

\[
\frac{1}{1 - z^4} = -\frac{1}{4} \left( \frac{1}{z - 1} \right) + \frac{1}{4} \left( \frac{1}{z + 1} \right) + \frac{1}{4i} \left( \frac{1}{z + i} \right) - \frac{1}{4i} \left( \frac{1}{z - i} \right)
\]

so that all we have to do is verify that functions of the form \( g(z) = 1/(z - z_0) \) satisfy the C-R Equations for any complex number \( z_0 = x_0 + iy_0 \).

To see this, note that

\[
\frac{1}{z - z_0} = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} + i \frac{-(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} = u + iv.
\]

Therefore we have that

\[
u_x = \frac{(y - y_0)^2 - (x - x_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2}, \quad u_y = \frac{-2(x - x_0)(y - y_0)}{((x - x_0)^2 + (y - y_0)^2)^2}
\]

and

\[
v_x = \frac{2(x - x_0)(y - y_0)}{((x - x_0)^2 + (y - y_0)^2)^2}, \quad v_y = \frac{(y - y_0)^2 - (x - x_0)^2}{((x - x_0)^2 + (y - y_0)^2)^2}.
\]

Hence we see that for each term in the partial fraction expansion, \( u_x = v_y \) and \( u_y = -v_x \) as required.

Exercise 13.4.14.
Solution:

We need to verify that the function \( v = -y/(x^2 + y^2) \) is harmonic. To simplify the calculation, let \( r = x^2 + y^2 \). Then \( r_x = 2x \) and \( r_y = 2y \). Therefore,

\[
v_x = \frac{0 - (-yr_x)}{r^2} = \frac{2xy}{r^2}
\]

and

\[
v_{xx} = \frac{r^2(2y) - (2xy)(2rr_x)}{r^4} = \frac{2yr - 8x^2y}{r^3}.
\]
Also, 
\[ v_y = \frac{-r - (-yr_y)}{r^2} = \frac{-r + 2y^2}{r^2} \]
and 
\[ v_{yy} = \frac{(r^2)(-ry + 4y) - (2rr_y)(-r + 2y^2)}{r^4} = \frac{2yr + 4y^3 - 8y^3}{r^3} = \frac{6yr - 8y^3}{r^3} \]
Therefore, 
\[ v_{xx} + v_{yy} = \frac{2yr - 8x^2y + 6yr - 8y^3}{r^3} = \frac{8y(r - x^2 - y^2)}{r^3} = 0. \]
In order to find a harmonic conjugate of \( v \) we must solve \( u_x = v_y \) and \( u_y = -v_x \). Thus we have \( u_x = (r + 2y^2)/r^2 \). If we try a function \( u \) of the form \( u = f(x, y)/r \) then we arrive at \( u = x/r + h(y) \) as a general solution. Thus \( u_y = -2xy/r^2 + h'(y) \) and setting this equal to \( -v_x = -2xy/r^2 \) gives us that \( h(y) = \text{const} \) and we can take \( h(y) = 0 \). Therefore, \( u = x/(x^2 + y^2) \) is a harmonic conjugate of \( v \). Hence \( f(z) = u + iv = (x - iy)/(x^2 + y^2) = 1/z \) is the corresponding analytic function.

Exercise 13.4.20.
Solution:
We need to verify that the function \( u = \cos x \cosh y \) is harmonic.
\[ u_x = -\sin x \cosh y, \quad \text{and} \quad u_{xx} = -\cos x \cosh y. \]
Also, 
\[ u_y = \cos x \sinh y, \quad \text{and} \quad u_{yy} = \cos x \cosh y \]
so that clearly \( u_{xx} + u_{yy} = 0. \)
In order to find a harmonic conjugate of \( u \) we must solve \( v_y = u_x \) and \( v_x = -u_y \). Thus we have \( v_y = -\sin x \cosh y \) so that \( v = -\sin x \sinh y + h(x) \). Then \( v_x = -\cos x \sinh y + h'(x) \) and setting this equal to \( -u_y = -\cos x \sinh y \) gives us that \( h(x) = \text{const} \) and we can take \( h(x) = 0 \). Therefore, \( v = -\sin x \sinh y \) is a harmonic conjugate of \( u \). Hence 
\[ f(z) = u + iv = \cos x \cosh y - i \sin x \sinh y \]
\[ = \frac{1}{2}(\cos x(e^y + e^{-y}) - i \sin x(e^y - e^{-y})) \]
\[ = \frac{1}{2}(e^y(\cos x - i \sin x) + e^{-y}(\cos x + i \sin x)) \]
\[ = \frac{1}{2}(e^y e^{-ix} + e^{-y} e^{ix}) \]
\[ = \frac{1}{2}(e^{-i(x+iy)} + e^{i(x+iy)}) \]
\[ = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos(z) \]
is the corresponding analytic function.
Exercise 13.5.6.
Solution:
\[ e^{(1+i)i\pi} = e^{\pi + i\pi} = e^\pi (\cos \pi + i \sin \pi) = -e^\pi. \]

Exercise 13.5.16.
Solution:
\[ 3 + 4i = 5((3/5) + i(4/5)) = 5(\cos(.92730) + i \sin(.92730)). \]

Exercise 13.5.18.
Solution:
\[ e^{3z} = e^{3x + i3y} = e^{3x}(\cos(3y) + i \sin(3y)) = 4. \] Therefore we need \( e^{3x} = 4, \) \( \cos(3y) = 1 \) and \( \sin(3y) = 0. \) This means \( x = (1/3) \ln(4) \) and \( 3y = 2\pi n \) for \( n \in \mathbb{Z}, \) so that \( y = 2n\pi/3, \) \( n \in \mathbb{Z}. \) Hence the solutions to the problem are \( z = (1/3) \ln(4) + 2n\pi i/3, \) \( n \in \mathbb{Z}. \)

Exercise 14.1.2.
Solution:
\[ z(t) = 5 - 2it, \ -3 \leq t \leq 3 \] parametrizes the straight line segment from \( 5 + 6i \) to \( 5 - 6i. \)

Exercise 14.2.4.
Solution:
Since \( \bar{z} \) is not analytic, Cauchy’s Theorem does not apply. The curve is parametrized by \( z(t) = \cos t + i \sin t, \ 0 \leq t \leq 2\pi. \) Hence \( dz/dt = -\sin t + i \cos t \) and
\[ \int_C \frac{1}{\bar{z}} \, dz = \int_0^{2\pi} \frac{-\sin t + i \cos t}{\cos t - i \sin t} \, dt = \int_0^{2\pi} \frac{-2 \sin t \cos t + i(\cos^2 t - \sin^2 t)}{\cos t - i \sin t} \, dt \]
\[ = \frac{\cos^2 t}{0} + i(1/2) \sin t|_0^{2\pi} \]
\[ = (1 - 1) + i(0 - 0) = 0 \]
\[ z(t) = 5 - 2it, \ -3 \leq t \leq 3 \] parametrizes the straight line segment from \( 5 + 6i \) to \( 5 - 6i. \)

Exercise 14.2.22.
Solution:
We can decompose the integrand using partial fractions into
\[
\frac{7z - 6}{z(z - 2)} = \frac{3}{z} + \frac{4}{z - 2}.
\]

Therefore,
\[
\int_C \frac{7z - 6}{z^2 - 2z} \, dz = 3 \int_C \frac{1}{z} \, dz + 4 \int_C \frac{1}{z - 2} \, dz.
\]

Since \( C \) is a simple closed curve containing 0, the deformation principle says that \( \int_C \frac{1}{z} \, dz \) is the same as integrating \( 1/z \) around a circle centered at 0. Hence the first integral in the sum is \( 3(2\pi i) = 6\pi i \). Similarly the second integral has the same value as though it were integrated around a circle centered at 2. Hence its value is also \( 2\pi i \) so the second integral is \( 4(2\pi i) = 8\pi i \). Hence we arrive at
\[
\int_C \frac{7z - 6}{z^2 - 2z} \, dz = 14\pi i.
\]