Exercise 10.7.4.
Solution:

\[
\iiint_T e^{-(x+y+z)} \, dV = \int_0^2 \int_{2-x}^{2-x-y} \int_0^{2-x-y} e^{-(x+y+z)} \, dz \, dy \, dx
\]
\[
= \int_0^2 \int_0^{2-x} -e^{-(x+y+z)} \big|_0^{2-x-y} \, dy \, dx
\]
\[
= \int_0^2 \int_0^{2-x} e^{-2} + e^{-(x+y)} \, dy \, dx
\]
\[
= \int_0^2 (x-2)e^{-2} - (e^{-(x+y)}) \big|_0^{2-x} \, dx
\]
\[
= \int_0^2 (x-3)e^{-2} + e^{-x} \, dx
\]
\[
= \frac{1}{2}(x-3)^2e^{-2} - e^{-x} \big|_0^2
\]
\[
= \frac{1}{2}e^{-2} - e^{-2} - \frac{9}{2}e^{-2} + 1 = -5e^{-2} + 1.
\]

Exercise 10.7.20.
Solution:

\[
\mathbf{F} = [3xy^2, yx^3 - y^3, 3zx^2]\] so that \(\text{div} \, \mathbf{F} = 3y^2 + x^2 - 3y^2 + 3x^2 = 4x^2\). Also, \(T\) is the cylinder of radius 5 and height 2. Therefore,

\[
\iiint_T \text{div} \, \mathbf{F} \, dV = \iiint_D \int_0^2 4x^2 \, dz \, dA
\]

where \(D\) is the disk of radius 5 centered at the origin. Switching to polar coordinates for the outside integral gives \(4x^2 = 4r^2 \cos^2 \theta\) and \(dA = r \, dr \, d\theta\). Hence we arrive at

\[
\iiint_D \int_0^2 4x^2 \, dz \, dA = \int_0^{2\pi} \int_0^5 \int_0^{2\pi} 4r^2 \cos^2 \theta \, dz \, dr \, d\theta
\]
\[
= 2 \int_0^{2\pi} \int_0^5 4r^3 \cos^2 \theta \, dr \, d\theta
\]
\[
= 2 \int_0^{2\pi} r^4 \cos^2 \theta \bigg|_0^5 \, d\theta
\]
\[
= 1250 \int_0^2 \cos^2 \theta \, d\theta
\]
\[
= 1250 \left( \frac{\theta}{2} + \frac{1}{2} \sin(2\theta) \right) \bigg|_0^{2\pi} = 1250\pi.
\]
Exercise 10.8.2.

Solution:

Theorem 1 says that \( \iint_S \frac{\partial f}{\partial n} \, dA = 0 \) if \( \nabla^2 f = 0 \). For \( f = y^2 - x^2 \), \( \nabla^2 f = 0 \), and \( \nabla f = [-2x, -2y, 0] \). A parametrization for the outside part of the cylinder is \( r(u, v) = [\cos u, \sin u, v] \), \( 0 \leq u \leq 2\pi \), \( 0 \leq v \leq 5 \). Computing \( r_u \times r_v \) gives \( r_u \times r_v = [\cos u, \sin u, 0] \) so that \( |r_u \times r_v| = 1 \). Finally, we have that \( \frac{\partial f}{\partial n} = \nabla f \cdot n = [-2 \cos u, -2 \sin u, 0] \cdot [\cos u, \sin u, 0] = 2(\sin^2 u - \cos^2 u) = 2 \cos(2u) \).

Therefore, with \( S' \) the outside surface of the cylinder,

\[
\iint_{S'} \frac{\partial f}{\partial n} \, dA = \int_0^{2\pi} \int_0^5 2 \cos(2u) \, dv \, du = 5 \int_0^{2\pi} 2 \cos(2u) \, du = 0.
\]

The top and bottom of the cylinder have normal vector \( \mathbf{k} \) and \( -\mathbf{k} \). Hence for these surfaces \( \nabla f \cdot \mathbf{n} = 0 \) since \( \nabla f \) has zero \( \mathbf{k} \) component. Therefore the contribution to the surface integral from the top and bottom is zero. Therefore, \( \iint_S \frac{\partial f}{\partial n} \, dA = 0 \) and Theorem 1 is verified.

Exercise 10.8.4.

Solution:

With \( f = x \) and \( g = y^2 + z^2 \), \( \nabla^2 g = 4 \), \( \nabla f = [1, 0, 0] \) and \( \nabla g = [0, 2y, 2z] \). Hence \( \nabla f \cdot \nabla g = 0 \) and \( f \nabla^2 g + \nabla f \cdot \nabla g = 4x \). Therefore,

\[
\iiint_T (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV = \int_0^1 \int_0^2 \int_0^3 4x \, dz \, dy \, dx = \int_0^1 24x \, dx = 12.
\]

Green’s First Formula says that \( \iint_S \frac{\partial g}{\partial n} \, dA = 12 \). To verify this, note that \( S \) is the surface of the box \( T \) and hence is bounded by 6 planes parallel to the coordinate planes. By facing the box looking down the positive \( x \) axis toward the origin, we have the following.

For the front and back sides, \( x = 1 \), and \( n = \mathbf{i} \) or \( x = 0 \) and \( n = -\mathbf{i} \). In either case, \( \nabla g \cdot \mathbf{n} = [0, 2y, 2z] \cdot [\pm 1, 0, 0] = 0 \). Hence the contribution to the surface integral of the front and back sides of the box is zero.

For the left side, \( y = 0 \), \( \mathbf{n} = -\mathbf{j} \), so that \( \nabla g \cdot \mathbf{n} = -2y = 0 \), and for the bottom, \( z = 0 \), \( \mathbf{n} = -\mathbf{k} \), so that \( \nabla g \cdot \mathbf{n} = -2z = 0 \). Hence the contribution to the surface integral from the left side and the bottom is zero.
For the right side, \( y = 2, \ n = \mathbf{j} \) so that \( \nabla g \cdot \mathbf{n} = 2y = 4 \) and hence \( f(\nabla g \cdot \mathbf{n}) = 4x \). The contribution to the surface integral from this side is

\[
\int_0^1 \int_0^3 4x \, dz \, dx = \int_0^1 12x \, dx = 6.
\]

For the top of the box, \( z = 3, \ n = \mathbf{k} \) so that \( \nabla g \cdot \mathbf{n} = 2z = 6 \), and so \( f(\nabla g \cdot \mathbf{n}) = 6x \). The contribution to the surface integral from the top of the box is

\[
\int_0^1 \int_0^2 6x \, dy \, dx = \int_0^1 12x \, dx = 6.
\]

Therefore,

\[
\iint_S f \left( \frac{\partial g}{\partial n} \right) \, dA = 12
\]
as required.

Exercise 10.8.6.

**Solution:**

With \( f = x^4 \) and \( g = y^2 \), \( \nabla^2 f = 12x^2 \), and \( \nabla^2 g = 2 \). Hence \( \nabla^2 g - g \nabla^2 f = 2x^4 - 12x^2 y^2 \). Therefore,

\[
\iiint_T (f \nabla^2 g - g \nabla^2 f) \, dV = \int_0^1 \int_0^1 \int_0^1 2x^4 - 12x^2 y^2 \, dz \, dy \, dx
\]

\[
= \int_0^1 \int_0^1 2x^4 - 12x^2 y^2 \, dy \, dx
\]

\[
= \int_0^1 \left( 2x^4 y - 4x^2 y^3 \right) \bigg|_0^1 \, dx
\]

\[
= \int_0^1 2x^4 - 4x^2 \, dx
\]

\[
= \frac{2}{5} x^5 - \frac{4}{3} x^3 \bigg|_0^1 = \frac{-14}{15}.
\]

Green’s Second Formula says that \( \iint_S \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, dA = \frac{-14}{15} \). To verify this, note first that \( \nabla f = [4x^3, 0, 0] \), and \( \nabla g = [0, 2y, 0] \) and that \( S \) is the surface of the box bounded by 6 planes parallel to the coordinate planes. By facing the box looking down the positive \( x \) axis toward the origin, we have the following.

For the front side, \( x = 1, \) and \( n = \mathbf{i} \), so that

\[
f(\nabla g \cdot \mathbf{n}) - g(\nabla f \cdot \mathbf{n}) = (x^4)([0, 2y, 0] \cdot [1, 0, 0]) - (y^2)([4x^3, 0, 0] \cdot [1, 0, 0]) = 0 - (y^2)(4) = -4y^2.
\]

Hence the contribution from this side is

\[
\int_0^1 \int_0^1 -4y^2 \, dz \, dy = \int_0^1 -4y^2 \, dy = -\frac{4}{3}.
\]
For the back side, $x = 0$ and $n = -i$, so that
\[ f(\nabla g \cdot n) - g(\nabla f \cdot n) = (x^4)([0, 2y, 0] \cdot [-1, 0, 0]) - (y^2)([4x^3, 0, 0] \cdot [-1, 0, 0]) = 0 - (y^2)(0) = 0 \]
and there is no contribution from this side.

For the left side, $y = 0$, $n = -j$, so that
\[ f(\nabla g \cdot n) - g(\nabla f \cdot n) = (x^4)([0, 2y, 0] \cdot [0, -1, 0]) - (y^2)([4x^3, 0, 0] \cdot [0, -1, 0]) = 0. \]
and there is no contribution from this side. For the right side, $y = 1$ and $n = j$ so that
\[ f(\nabla g \cdot n) - g(\nabla f \cdot n) = (x^4)([0, 2y, 0] \cdot [0, 1, 0]) - (y^2)([4x^3, 0, 0] \cdot [0, 1, 0]) = 2x^4y - 0 = 2x^4. \]
Hence the contribution from this side is
\[ \int_0^1 \int_0^1 2x^4 \, dz \, dx = \int_0^1 2x^4 \, dx = \frac{2}{5}. \]

For the top and bottom, we have $z = 0$, $n = -k$ and $z = 1$, $n = k$. In both cases, $\nabla f \cdot n = [4x^3, 0, 0] \cdot [0, 0, \pm 1] = 0$ and $\nabla g \cdot n = [0, 2y, 0] \cdot [0, 0, \pm 1] = 0$, so there is no contribution to the integral from these sides.

\[ \iint_S \left( f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) \, dA = \frac{2}{5} - \frac{4}{3} = -\frac{14}{15} \]
as required.

Exercise 10.9.8.

Solution:

With $\mathbf{F} = [y^3, -x^3, 0],$
\[ \text{curl} \mathbf{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 & -x^3 & 0 \end{vmatrix} = (0)i - (0)j + (-3(x^2 + y^2))k. \]

For the surface $S: x^2 + y^2 \leq 1, z = 0$, the unit normal vector is $n = \pm k$. For this calculation, we will choose $n = k$. Hence $\mathbf{F} \cdot n = -3(x^2 + y^2)$. Parametrizing $S$ using polar coordinates gives
\[ \iint_S -3(x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^1 -3r^2 \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 -3r^3 \, dr \, d\theta = (2\pi) \left( -\frac{3}{4} \right) = -\frac{3\pi}{2}. \]
Exercise 10.9.10.

Solution:

In order to verify Stokes’s Theorem for the above example, we note that the boundary curve $S$ for the surface $S$ is the unit circle which can be parametrized by $\mathbf{r}(t) = [\cos t, \sin t, 0]$ $0 \leq t \leq 2\pi$, and that $\mathbf{r}'(t) = [-\sin t, \cos t, 0]$ and that $\mathbf{F}(\mathbf{r}(t)) = [\sin^3 t, -\cos^3 t, 0]$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [\sin^3 t, -\cos^3 t, 0] \cdot [-\sin t, \cos t, 0] \, dt$$

$$= \int_0^{2\pi} -(\sin^4 t + \cos^4 t) \, dt$$

$$= -\left( \frac{1}{4} \cos^3 t \sin t - \frac{1}{4} \sin^3 t \cos t + \frac{3}{4} t \right)_{0}^{2\pi}$$

$$= 3\pi$$

$$= \frac{3\pi}{2}$$

as required.